Weyl focusing effects on image magnification due to randomly distributed isothermal objects

Takashi Hamana

Astronomical Institute, Tohoku University, Sendai 980-77, Japan

ABSTRACT

Weyl focusing effects on image magnifications are investigated by using the multiple gravitational lens theory. We focus on the gravitational lensing effects that are a result of small-scale virialized objects, such as galaxies and clusters of galaxies. We consider a simple model of an inhomogeneous universe. The matter distribution in the universe is modelled by randomly distributed isothermal objects. We found that, for the majority of the random lines of sight, the Weyl focusing has no significant effect and the image magnification of a point-like source within a redshift of 5 is dominated by Ricci focusing.

Key words: cosmology: miscellaneous – gravitational lensing.

1 INTRODUCTION

The distance–redshift relation plays an important role in astronomy, and specifically in observational cosmology. The standard distance, which has been used in most previous studies, is based on a postulate that a distribution of matter in the universe is homogeneous (e.g. Weinberg 1972). It has, however, been well recognized that our Universe is highly inhomogeneous on small-scales. Since the inhomogeneities of the mass distribution focus (defocus) the bundle of light rays (the gravitational lensing effects), the distance in an inhomogeneous universe deviates from that in the homogeneous Friedmann universe. It is, therefore, obvious that a detailed understanding of the propagation of light rays in the inhomogeneous universe is necessary for correct studies of objects in a distant universe.

Since the pioneering work by Gunn (1967), there has been a lot of progress in this subject. Babul & Lee (1991), among others, studied the effects of Ricci focusing by weak inhomogeneities. They found that the dispersion in image magnifications due to large-scale virialized objects, such as galaxies and clusters of galaxies, have not been fairly taken into account. Kayser & Refsdal (1988) investigated the gravitational lensing effects arising from randomly distributed King model galaxies. They paid special attention to a high-magnification part of the magnification probability distribution. Recently Wambsganss, Cen & Ostriker (1998) studied the gravitational lensing effects by using large N-body simulation with an effective resolution of comoving $10h^{-1} \text{Mpc}$. First, they shoot the light ray through the lens planes by using the multiple gravitational lens equation, then the magnification matrix is determined from the mapping of the light ray positions between the image and the source plane. In this procedure, Ricci and Weyl focusing cannot be treated independently, therefore no discussion is given for the Weyl focusing effect in their paper. However, the magnification factor as a function of position in the source plane and image plane are presented in figs 4 and 6 of their paper. Those figures show that there is quite a large region that is demagnified by a small amount, and a few relatively small spots that are quite highly magnified by the small-scale inhomogeneities.

The purpose of this paper is to examine the Weyl focusing effect arising from the small-scale inhomogeneities; we are not concerned with the effect resulting from the large-scale structure. The density distribution in the universe is modelled as a randomly distributed isothermal lenses. The isothermal lens is approximated by virialized objects, such as a galaxy and a cluster of galaxies. This model is similar to the one studied by Kayser & Refsdal (1988). However, our emphasis is different from theirs, i.e. we pay special attention to the gravitational lensing effects on a majority of light rays. Although this model is a very simplified and unrealistic one, we believe that the model is good enough to allow us to investigate the essential points of the Weyl focusing effect arising from the small-scale inhomogeneities.

This paper is organized as follows: in Section 2, we describe our
2 MODELS AND METHOD

2.1 Theory of multiple gravitational lensing

We use multiple lens equations to trace the propagation of infinitesimal bundles of light rays. Schneider, Ehlers & Falco (1992) deal with the theory of multiple gravitational lensing in detail. Here we simply describe only those aspects that are directly relevant to this paper.

As was done in previous studies (see, e.g., Blandford & Narayan 1986, Kovner 1987), we consider \( N \) screens (lens planes) between an observer \((z = 0)\) and source \((z_i)\) located at redshifts \( z_i \) with \( 1 \leq i \leq N + 1 \). In the following, the quantities on the lens and source planes are described by indices \( A, B, \ldots = \{ 1, 2 \} \). The position vector of a light ray on the \( i \)-th lens plane is denoted by \( y_i(z_i) \). Let \( \alpha_i \) denote the deflection angle of a light ray at position \( y_i(z_i) \) on the \( i \)-th lens plane. The multiple gravitational lens equation and the evolution equation of the lensing magnification matrix \( M_{AB} \) are written as

\[
y_A(z_j) = \frac{D_j}{D_i} y_A(z_i) - \sum_{i=1}^{N} D_i \frac{D_i}{D_j} \alpha_i \left[ y_A(z_i) \right],
\]

\[
M_{AB}[y(z_j)] = \delta_{AB} - \sum_{i=1}^{N} D_i \frac{D_i}{D_j} \alpha_i \left[ y_A(z_i) \right] M_{CB}[y(z_i)].
\]

for \( 2 \leq j \leq N + 1 \), where \( D_i \) (\( D_j \)) denotes the standard angular diameter distance between redshifts of \( z_i \) and \( z_j \) (0 and \( z_i \)) with \( i < j \), the comma denotes differentiation with respect to the components of \( y_A(z_i) \) and \( y_A(z_i) = y_A(z_i) \). The deflection angle \( \alpha_i \) is determined by the equation

\[
\hat{\alpha}_A(y) = \frac{4G}{c^2} \int d^2y' \frac{-y - y'}{\left| y - y' \right|^2} \left( \Sigma(y') - \langle \Sigma \rangle \right),
\]

where \( \Sigma(y) \) is the surface mass density and \( \langle \Sigma \rangle \) is its average value. For convenience in notation, we introduce the following quantities:

\[
\alpha_i(z_i, z_j) = D_i \frac{D_i}{D_j} \hat{\alpha}_A[y_A(z_i)],
\]

\[
T_{AB}(z_i, z_j) = D_i \frac{D_i}{D_j} \alpha_{AB}[y(z_i)].
\]

The optical tidal matrix \( T_{AB}(z_i, z_j) \) is decomposed into the Ricci and Weyl focusing terms, respectively

\[
R(z_i, z_j) = \frac{1}{2} \left[ T_{11}(z_i, z_j) + T_{22}(z_i, z_j) \right],
\]

\[
\mathcal{F}(z_i, z_j) = \frac{1}{2} \left[ T_{11}(z_i, z_j) - T_{22}(z_i, z_j) \right] + iT_{12}(z_i, z_j).
\]

In general, equation (2) is not an explicit equation for \( M_{AB} \), since the equation involves a summation over \( T_{AB} \) evaluated on the light ray path, such that one first has to solve the multiple gravitational lens equation (1). However, for the light rays travelling in regions where \( \alpha_i \) and \( T_{AB} \) \( \ll 1 \), one can expand \( M_{AB} \) in powers of \( \alpha_i \) and \( T_{AB} \) about its value when the light ray is unperturbed. We rewrite equation (1) as

\[
y_A(z_j) = y_A^{(0)}(z_j) + y_A^{(1)}(z_j) + O(\alpha^2)
\]

and equation (2) as

\[
M_{AB}[y(z_j)] = M_{AB}^{(0)}[y^{(0)}(z_j)] + M_{AB}^{(1)}[y^{(0)}(z_j)] + M_{AB}^{(2)}[y^{(0)}(z_j)] + \ldots,
\]

where \( y_A^{(0)}(z_j) \) is the first term of the right-hand side of equation (1) and \( y_A^{(1)}(z_j) \) is the second term, but the deflection angle is evaluated at the unperturbed light ray position. Expanding equation (2) in terms of \( \alpha_A \) and \( T_{AB} \), one finds

\[
M_{AB}^{(0)}(z_j) = \delta_{AB},
\]

\[
M_{AB}^{(1)}(z_j) = \sum_{i=1}^{N} T_{AB}(z_i, z_j),
\]

\[
M_{AB}^{(2)}(z_j) = \sum_{k=1}^{N-1} \sum_{i=1}^{N-1} \left[ T_{AC}(z_i, z_j) T_{CB}(z_k, z_j) + T_{ABC}(z_i, z_j) \alpha_C(z_k, z_j) \right],
\]

for \( 3 \leq j \leq N + 1 \). In the above expressions, \( T_{AB} \) and \( \alpha_A \) are evaluated at the unperturbed light ray position. The image magnification factor of a point-like source is given by the inverse of the determinant of the magnification matrix, i.e.,

\[
\mu = \left| \det M_{AB}^{(2)} \right|^{-1}.
\]

Up to the order of \( M_{AB}^{(2)} \), the determinant is

\[
\det M_{AB}(z_j) = 1 - 2 \sum_{i=1}^{N} \mathcal{R}(z_i, z_j) + \left[ \sum_{i=1}^{N} \mathcal{R}(z_i, z_j) \right]^2
\]

\[+ \Re \langle \mathcal{F}^*(z_i, z_j) \mathcal{F}(z_i, z_j) \rangle + \mathcal{R}(z_i, z_j) \alpha_A(z_i, z_j) \},
\]

for \( 3 \leq j \leq N + 1 \). This is our principal equation. In general, the effects of Ricci and Weyl focusing on image magnification are coupled. However, up to this order, they are not coupled. We call the terms in equation (11) which involve Ricci focusing terms the ‘Ricci contribution’ and those that involve Weyl focusing terms are called the ‘Weyl contribution’.

2.2 Truncated singular isothermal sphere lens model

We adopt the truncated singular isothermal sphere as the lens model. Its surface mass density is written as

\[
\Sigma(R) = \frac{\sigma_v^2}{2GR} \left( 1 + \frac{R}{R_G} \right)^{-2},
\]

where \( \sigma_v \) is the one-dimensional velocity dispersion and \( R_G \) is the half-mass radius (Pei 1993). The corresponding 3D mass density runs as \( \rho \propto R^{-2} \) (\( R \ll R_G \)) and as \( \rho \propto R^{-4} \) (\( R \gg R_G \)), and the mass within the radius \( R \) is

\[
m(R) = \frac{\pi \sigma_v^2 R_G}{G} \frac{R^2}{R_G},
\]

then the total mass is \( m_{tot} = \pi \sigma_v^2 R_G \). This model should provide a fair approximation to virialized objects, such as a galaxy and a cluster of galaxies with isothermal dark haloes.

For the distribution of matter in the universe, we assume that the isothermal lenses are randomly distributed with the average mass density \( \rho_L(z) \) and the rest of the matter has a uniform distribution. We also assume that the comoving density of lenses is constant in time, thus \( \rho_L(z) = (1 + z)^3 \rho_L(0) \). Furthermore, we approximate the lensing effects of the isothermal objects, except the nearest one, to a form of uniform surface mass density.

Under the above assumptions and the circularly symmetric mass
distribution of the lens model, the deflection angle is given by
\[
\alpha(z_i, z_j) = \sqrt{\alpha_i^2(z_i, z_j) + \alpha_j^2(z_i, z_j)}
\]
where \( R \) is the distance between a light ray position and the centre of the nearest lens object in the \( i \)-th lens plane, and \( \delta z \) is the redshift interval between \( (i - 1) \)-th and \( i \)-th lens planes, and \( \Omega_0 \) is the average surface mass density of the objects in the \( i \)-th lens plane. From the deflection angle (13), the Ricci and Weyl focusing terms are immediately given by
\[
F(z_i, z_j) = a_{ij}(z_i, z_j) \left[ \frac{1}{2} \frac{R_G^2}{2 R (R_G + R_j)} - \frac{R_G}{R_c (R_G + R_c)} \right],
\]
and
\[
F(z_i, z_j) = a_{ij}(z_i, z_j) \left[ \frac{1}{2} \frac{R_G^2}{2 R (R_G + R_j)} - \frac{R_G}{R_c (R_G + R_c)} \right].
\]
We introduce a compactness parameter \( \nu \) as follows;
\[
\nu = 4 \pi \left( \frac{\sigma_v}{c} \right)^2 \frac{H_0}{R_G},
\]
where \( \sigma_v \) is the velocity dispersion of the lens object (the inner region of \( \sim 0.5 \times \) Einstein radius), therefore the validity of the assumptions used in deriving equation (11) breaks down for light rays passing through that region. We now estimate the strong lensing effect on our study in two ways.

The first is based on an order-of-magnitude estimate (section 14 of Peebles 1993, and also Futamase & Sasaki 1989); we examine a magnitude of the tidal matrix \( T_{ij} \). The lensing objects are randomly distributed and each with mass \( M = 2 \sigma_v^2 R_G^2 \), where \( \sigma_v \) is a characteristic comoving size of a lens object and is of order \( R_G \).

Hence the mean comoving number density of the lens objects is \( n_i = \Omega_k (3 H_0^2 / 16 \pi) a_i^{-2} t^{-1} \), so the mean comoving separation distance is \( r_i = \Omega_k^{-1} (3 H_0^2 / 16 \pi) a_i^{-3} t^{1/3} \). Then, for a geodesic affine comoving distance of \( \lambda \), the light ray gravitationally encounters such objects \( N_s = \lambda r_i n_i \) times on average. At each encounter, the contribution to the tidal matrix is \( \delta T = 4 \pi (\sigma_v^{-1} c^{-1} r_i / b)^2 (D_E D_{\delta E} D_{E T} ) \sim 4 \pi (\sigma_v^{-1} c^{-1} r_i^2) (D_E D_{\delta E} D_{E T} ) \), where \( D_E \) is a comoving angular diameter distance and the subscript \( d \) stands for a lens (source), \( b \) is the comoving impact parameter, and we have assumed that the mean comoving impact parameter is of order \( r_i \). Since the sign of each contribution will be random, the total contribution to the tidal optical matrix will be
\[
\delta T \sqrt{N_s} \sim 3 \frac{4 \pi (\sigma_v^{-1} c^{-1})^2}{176 H_0} \left[ \frac{H_0}{D_E} \right] \left[ \frac{H_0}{D_{\delta E}} \right] \left[ \frac{H_0}{D_{ET}} \right] \left[ \frac{H_0}{c} \right] \left[ \frac{a_i}{t} \right] \left[ \frac{a_i}{t} \right] .
\]
The contribution from the direct encounters can be similarly estimated as
\[
\delta T \sim 3 \frac{4 \pi (\sigma_v^{-1} c^{-1})^2}{176 H_0} \left[ \frac{H_0}{D_E} \right] \left[ \frac{H_0}{D_{\delta E}} \right] \left[ \frac{H_0}{D_{ET}} \right] \left[ \frac{H_0}{c} \right] \left[ \frac{a_i}{t} \right] \left[ \frac{a_i}{t} \right] .
\]
We numerically integrate the last equation for cases of the lens models with \( \Omega_k, \nu = (1, 0.1), (1, 1), (0.2, 0.1) \), and for the Einstein–de Sitter universe model. The results are presented in Table 1. From Table 1, it can be found that the probabilities of strong lensing events are very small except for an extreme model with \( \Omega_k, \nu = (1, 1) \). Even for the extreme model, the probability is not significantly large. We thus conclude that the strong lensing effects do not significantly alter our results. It can be said from the above two estimations that we can safely use the perturbative equation (11).

2.3 Ray shooting
Since we have assumed the random distribution for lens objects, the probability of finding lenses in some region on a lens plane is
The lensing effects are mainly a result of the nearest lens and are well approximated by equations (13), (16) and (17). Then all the necessary information about evaluating the magnification factor (11) is obtained by randomly determining the relative position between a light ray and the nearest lens object in each lens plane. We perform Monte Carlo simulations to trace the propagation of light rays. The procedure is as follows.

(i) First of all, we determine the redshift intervals of lens planes to satisfy the condition that \( R_G/R_c < 0.1 \).

(ii) The relative positions between a light ray and a centre-of-lens object are randomly determined in each lens plane.

(iii) Summations in equation (11) are performed for each term, and the results are stored in a file.

Steps (ii) and (iii) are repeated for each light ray.

3 RESULTS

For the background universe, we only consider the Einstein–de Sitter universe model, i.e., \( \Omega_0 = 1 \) and \( \lambda_0 = 0 \). For the lens models, we choose the compactness parameter \( n \) which roughly correspond to a galaxy-scale inhomogeneity and the scale of a cluster of galaxies, respectively. The density parameter of the lens objects are set to be \( \Omega_L = 1 \) and \( 0 < \Omega_L < 1 \). From the condition \( R_G/R_c < 0.1 \), the redshift intervals between lens planes are typically set to be \( \sim 10^{-2} \). 10^6 runs are performed for each model. For each run (light ray), we then obtain the Ricci and Weyl contribution and also the image magnification factor. This immediately gives distribution functions of runs on the Ricci–Weyl contribution plane for all the source redshifts. We calculate number densities of results of runs in the Ricci–Weyl contribution plane. The peaks of the number density

![Figure 1](image_url)
and the isodensity contours which enclose 68 per cent of all runs are presented in Fig. 1. The probability distributions of the Ricci and Weyl contributions for the case of the source redshift of \( z_s = 3 \) in \((\Omega_L, \nu) = (1, 1)\) model are shown in Fig. 2. As is clearly shown in the optical scalar equation (see, e.g., Schneider et al. 1992), the Weyl contribution is always negative. Fig. 1 reveals that the Weyl contributions in the majority of light rays are rather small, except in the case where the source redshifts \( z_s \geq 3 \) in the \((\Omega_L, \nu) = (1, 1)\) model. Alternatively, this point can also be shown in Fig. 2. The probability distribution of the Weyl contribution has a narrow peak centred at very small values, in marked contrast with that of the Ricci contribution which has a broad distribution. Since we have assumed no evolution for lens objects, the dispersion keeps on spreading in the Ricci–Weyl contribution plane even for high redshift.

In order to examine the effects of the Weyl focusing on the image magnification quantitatively, we calculate the magnification factors evaluated without the Weyl contribution (denoted by \( \mu_R \)). Then we calculate the following quantity which measures the influence of Weyl contribution on the image magnifications:

\[
\Delta \mu = \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left( 1 - \frac{\mu_i}{\mu} \right)^2 },
\]

where the summation is taken only over rays with \( \det M_{AB} > 0 \). The rays with \( \det M_{AB} < 0 \) belong to a multiple image system, therefore the percentage of runs which are excluded in the above evaluation roughly exhibits the probability of the strong lensing events among random lines of sight. At the same time, the above condition also excludes the rays which pass through the high \( T_{AB} \) region. The results are presented in Table 2 with the percentages of the excluded runs (in parentheses).

Observationally, the strong lensing events among the random lines of sight are very rare. For example, the probability of the multiply imaged quasars in a quasar sample is at most \( 10^{-2} \) (e.g. Claeskens, Jaunsen & Surdej 1996). Combining this fact with our results summarized in Table 2, it may be reasonably concluded that,

---

Figure 2. Probability distributions for a model with \((\Omega_L; \nu) = (1; 1)\) and \( z_s = 3 \). (a) Isodensity contours which enclose 38, 68 and 87 per cent (inner to outer) of all runs. The plus denotes the peak of the surface number density of runs. (b) The probability distribution of Ricci contributions. (c) The probability distribution of Weyl contributions.
as far as our simple matter distribution model is concerned, the Weyl focusing has a negligible effect on the light rays even for sources at a redshift of 5. This result can be naturally explained by the following two reasons: first, the Weyl focusing is a second-order effect on the image magnification, therefore it becomes important only for the light rays passing through a very high non-linear (relatively rare) region. Secondly, since we have assumed a random distribution for the isothermal lenses, the rays coherently affected by the Weyl focusing are very rare, consequently, the majority of light rays are only weakly influenced by the Weyl focusing.

4 DISCUSSIONS

In this paper, we restricted our study to the gravitational lensing effects due to randomly distributed isothermal lenses. We found that the Weyl focusing effect is small, $|\Delta \mu| \lesssim 0.1$, for a majority of light rays.

Lee & Paczynski (1990) examined the gravitational lensing effects in randomly distributed clumps with Gaussian surface mass density profile. They found that the image magnifications are dominated by Ricci focusing and the Weyl focusing has no significant effect. They only considered the case of a source redshift of $z_s = 1.631$, in an $\Omega_0 = 1$ and $\lambda_0 = 0$ universe. However, we have found that, as far as the random distribution of lens objects is concerned, the Weyl focusing is rather small even for higher redshifts.

In this study, a correlation of lens objects and large-scale structure is not taken into account. The study of the influences of the correlation on the Weyl focusing lies outside the scope of this paper, and will be examined in future works. On the other hand, the Weyl focusing effect due to large-scale structures is investigated by using N-body simulation (Jaroszyński et al. 1990) and analytically (Nakamura 1997). These two studies show that, although there is an uncertainty in the normalization of the density power spectrum, the magnitude of Weyl focusing arising from the large-scale structure is of the order $10^{-2} \sim 10^{-1}$ for the source redshifts of $1 \lesssim z_s < 5$ (fig. 3 of Jaroszyński et al. 1990, fig. 3 of Nakamura 1997). Consequently it can be stated that, comparing the results of the above-mentioned studies with our results, the Weyl focusing effect due to large-scale structures is comparable with, or larger than, that due to the small-scale inhomogeneities.

ACKNOWLEDGMENTS

The author wishes to thank the referee for valuable comments on the first version of the manuscript which significantly improved the quality of this paper. TH would like to thank Professor P. Schneider, Professor T. Futamase, Dr. M. Hattori and M. Takada for fruitful discussions. TH would also like to thank Dr. P. Premadi for carefully reading and commenting on the manuscript.

REFERENCES


This paper has been typeset from a TeX/LATEX file prepared by the author.