Let Q be any quantity associated with each particle. The average value of Q is defined by

$$\langle Q \rangle = \frac{1}{n}\int d^3u\, Q f. \qquad (2.36)$$

where

$$n = \int d^3u\, f$$

the number dencity per unit volume.

It follows that

$$\int d^3u\, Q f = n\langle Q \rangle.$$

→ total amount of the quantity Q per unit volume.

Using the definition (2.36), the non-zero terms in (2.35) can be put in the form

$$\frac{\partial}{\partial t}(n\langle \chi \rangle) + \frac{\partial}{\partial x_i}(n\langle u_i \chi \rangle) - n\left\langle u_i \frac{\partial \chi}{\partial x_i}\right\rangle - \frac{n}{m}\left\langle F_i \frac{\partial \chi}{\partial u_i}\right\rangle - \frac{n}{m}\left\langle \frac{\partial F_i}{\partial u_i}\chi \right\rangle = 0 \quad (2.37)$$

→ how the volume density $n\langle \chi \rangle$ of any quantity χ conserved in binary collisions evolves in time.

P.30. § Exercises

○ 2.1.

When, $u_0 = 0$, the Maxwellian distribution (2.24) is

$$f(\mathbf{u}) = n\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{m u^2}{2k_B T}\right)$$

$$\left(\because \mathbf{u}^2 \to u^2,\ |\mathbf{u}| = u = \sqrt{u_x^2 + u_y^2 + u_z^2}\right)$$

then,

$$\langle u \rangle = \frac{1}{n}\int d^3u\, u\, f(u)$$

$$= \frac{1}{n}\int d^3u\, u\, n\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{m u^2}{2k_B T}\right)$$

$$= 4\pi \int_0^\infty du\, u^3 \exp\left(-\frac{m u^2}{2k_B T}\right)$$

$$\left(\begin{aligned}
\because \int_0^\infty dx\, x^3 \exp(-Ax^2) &= \left[x^2\left(-\frac{1}{2A}\right)\exp(-Ax^2)\right]_0^\infty + \frac{1}{2A}\int_0^\infty 2x \exp(-Ax^2)\,dx \\
&= \frac{1}{A}\int_0^\infty x \exp(-Ax^2)\,dx \\
&= \frac{1}{A}\left[\left(-\frac{1}{2A}\right)\exp(-Ax^2)\right]_0^\infty \\
&= -\frac{1}{2A^2}(0-1) = \frac{1}{2A^2}
\end{aligned}\right)$$

$$= 4\pi\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \cdot \frac{1}{2}\left(\frac{2k_B T}{m}\right)^2$$

$$= 2\pi \cdot \pi^{-\frac{3}{2}}\left(\frac{m}{2k_B T}\right)^{\frac{3}{2}} \cdot \left(\frac{2k_B T}{m}\right)^2$$

$$= 2\sqrt{\frac{2k_B T}{\pi m}} = \sqrt{\frac{8k_B T}{\pi m}}$$

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u_{rms}
根二乗平均速度.

The root mean square spead is given by

$$u_{rms} = \sqrt{\langle u^2 \rangle} = \sqrt{\frac{1}{n}\int d^3u\, u^2 f(u)}$$

$$\frac{1}{n}\int d^3u\, u^2 f(u)$$

$$= \frac{1}{n}\int d^3u \cdot u^2\, n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \exp\left(-\frac{mu^2}{2k_BT}\right)$$

$$= \left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \int d^3u\, u^2 \exp\left(-\frac{mu^2}{2k_BT}\right)$$

$$= 4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \int_0^\infty du\, u^4 \exp\left(-\frac{mu^2}{2k_BT}\right)$$

$$\left(\begin{aligned} \because \int_0^\infty dx\, x^4 \exp(-Ax^2) &= \left[x^3\cdot\left(-\frac{1}{2A}\right)\exp(-Ax^2)\right]_0^\infty + \frac{1}{2A}\int_0^\infty 3x^2 \exp(-Ax^2)\,dx \\ &= \frac{3}{2A}\left[x\left(-\frac{1}{2A}\right)\exp(-Ax^2)\right]_0^\infty + \frac{3}{4A^2}\int_0^\infty \exp(-Ax^2)\,dx \\ &= \frac{3}{4A^2}\cdot\frac{1}{2}\left(\frac{\pi}{A}\right)^{\frac{1}{2}} = \frac{3\pi^{\frac{1}{2}}}{8}\cdot A^{-\frac{5}{2}} \end{aligned}\right)$$

$$= 4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\cdot\frac{3\pi^{\frac{1}{2}}}{8}\cdot\left(\frac{2k_BT}{m}\right)^{\frac{5}{2}}$$

$$= \frac{3}{2}\cdot\frac{2k_BT}{m} = \frac{3k_BT}{m}$$

$$\therefore u_{rms} = \sqrt{\frac{3k_BT}{m}}$$

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○ 2.2

The number of collisions δn_c per unit volume per unit time is

$$\delta n_c = \sigma(u, u_1 | u', u_1')\cdot n\,|u - u_1|\, n_1\, d\Omega \quad (2.6)$$

We write the differential scattering cross-section as a function of the scattering angle Ω between u and u', so that

$$\delta n_c = \sigma(\Omega)\,|u - u_1|\, f(u)\, f(u_1)\, d\Omega\, d^3u\, d^3u_1$$

$$\left(\because\ n = f(u)\,d^3u,\quad n_1 = f(u_1)\,d^3u_1\right)$$

and then integrating over Ω and u and u_1, i.e.

$$N_{coll} = \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\,|u - u_1|\, f(u) f(u_1)$$

Now consider the molecules to be hard spheres of radius a so that the total scattering cross-section is πa^2, the above integral for such molecules is ~~to obtain~~

$$N_{coll} = \int d^3u \int d^3u_1 \int d\Omega\, \overset{\sigma(\Omega)}{\cancel{\pi a^2}}\, n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \exp\left(-\frac{mu^2}{2k_BT}\right) n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \exp\left(-\frac{mu_1^2}{2k_BT}\right)$$

$$= \pi a^2 n^2 \left(\frac{m}{2\pi k_BT}\right)^3 \int d^3u \int d^3u_1 \exp\left(-\frac{m(u^2+u_1^2)}{2k_BT}\right)$$

$$\left(\because\ u - u_0 \to u,\quad u_1 - u_0 \to u_1\right)$$

↳ we go to a frame moving with velocity u_0

Transform from u and u_1
to the relative velocity $u_{rel} = u - u_1$
and the center-of-mass velocity $u_{cm} = \frac{u + u_1}{2}$

$$u = \frac{u_{rel} + 2u_{cm}}{2}, \quad u_1 = \frac{-u_{rel} + 2u_{cm}}{2}$$

$$u^2 + u_1^2 = \left(\frac{u_{rel} + 2u_{cm}}{2}\right)^2 + \left(\frac{-u_{rel} + 2u_{cm}}{2}\right)^2$$

$$= \frac{u_{rel}^2 + 4u_{rel}u_{cm} + 4u_{cm}^2 + u_{rel}^2 - 4u_{rel}u_{cm} + 4u_{cm}^2}{4}$$

$$= \frac{u_{rel}^2}{2} + 2u_{cm}^2$$

Jacobian $\frac{\partial(u, u_1)}{\partial(u_{rel}, u_{cm})} = \begin{vmatrix} \frac{\partial u}{\partial u_{rel}} & \frac{\partial u}{\partial u_{cm}} \\ \frac{\partial u_1}{\partial u_{rel}} & \frac{\partial u_1}{\partial u_{cm}} \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$$

so that

$$N_{coll} = \pi n^2 a^2 \left(\frac{m}{2\pi k_B T}\right)^3 \int d^3u_{rel}\, u_{rel} \exp\left[-\frac{m u_{rel}^2}{4k_BT}\right] \int d^3u_{cm} \exp\left[-\frac{m u_{cm}^2}{k_BT}\right]$$

$$\because \int d^3u_{rel}\, u_{rel} \exp\left[-\frac{m u_{rel}^2}{4k_BT}\right]$$

$$= 4\pi \int_0^\infty du_{rel}\, u_{rel}^3 \exp\left[-\frac{m u_{rel}^2}{4k_BT}\right]$$

$$= 4\pi \cdot \frac{1}{2}\left(\frac{4k_BT}{m}\right)^2 = 2\pi\left(\frac{4k_BT}{m}\right)^2$$

$$\int d^3u_{cm}\, \cancel{u_{cm}} \exp\left[-\frac{m u_{cm}^2}{k_BT}\right]$$

$$= 4\pi \int_0^\infty du_{cm}\, u_{cm}^2 \exp\left[-\frac{m u_{cm}^2}{k_BT}\right]$$

$$\int_0^\infty dx\, x^2 \exp(-Ax^2)$$

$$= \left[x \cdot \left(-\frac{1}{2A}\right) \exp(-Ax^2)\right]_0^\infty + \frac{1}{2A}\int_0^\infty \exp(-Ax^2)\, dx$$

$$= \frac{1}{2A} \cdot \frac{1}{2}\left(\frac{\pi}{A}\right)^{\frac{1}{2}} = \frac{1}{4}\frac{\pi^{\frac{1}{2}}}{A^{\frac{3}{2}}}$$

$$= 4\pi \cdot \frac{1}{4}\pi^{\frac{1}{2}}\left(\frac{k_BT}{m}\right)^{\frac{3}{2}}$$

$$= \pi^{\frac{3}{2}}\left(\frac{k_BT}{m}\right)^{\frac{3}{2}}$$

$$N_{coll} = \pi n^2 a^2 \left(\frac{m}{2\pi k_BT}\right)^3 \cdot 2\pi\left(\frac{4k_BT}{m}\right)^2 \cdot \pi^{\frac{3}{2}}\left(\frac{k_BT}{m}\right)^{\frac{3}{2}}$$

$$= n^2a^2 \cdot \pi \cdot \frac{1}{8\pi^3} \cdot 2\pi \cdot 16 \cdot \pi^{\frac{3}{2}} \cdot \left(\frac{m}{k_BT}\right)^{3-2-\frac{3}{2}}$$

$$= 4n^2a^2\left(\frac{\pi k_BT}{m}\right)^{\frac{1}{2}}$$

The number of collisions taking place ~~ø~~ per unit time per unit volume per a particle is

$$\frac{N_{coll}}{n} = 4na^2\left(\frac{\pi k_BT}{m}\right)^{\frac{1}{2}}$$

then,

$$\lambda \times \frac{N_{coll}}{n} = \langle u \rangle.$$

($\because$ λ : the mean free path)

i.e.

$$\lambda = \langle u \rangle \times \frac{n}{N_{coll}}$$

$$= \sqrt{\frac{8k_BT}{\pi m}} \times \frac{1}{4na^2}\left(\frac{m}{\pi k_BT}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}\, n\pi a^2}$$

○ 2. 3.

The H function is

$$H = \int d^3u\, f \log f$$

so that

$$H = \int d^3u\, n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \exp\left(-\frac{mu^2}{2k_BT}\right)\left(\log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \exp\left(-\frac{mu^2}{2k_BT}\right)\right)$$

$$= n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \int d^3u\, \exp\left(-\frac{mu^2}{2k_BT}\right)\left[\log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} + \left(-\frac{mu^2}{2k_BT}\right)\right]$$

$$= 4\pi n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \int du \left[\log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} u^2 \exp\left(-\frac{mu^2}{2k_BT}\right)\right.$$
$$\left.\left(-\frac{m}{2k_BT}\right) u^4 \exp\left(-\frac{mu^2}{2k_BT}\right)\right]$$

$$= 4\pi n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \left[\log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \int d^3u\, u^2 \exp\left(-\frac{mu^2}{2k_BT}\right)\right.$$
$$\left. - \frac{m}{2k_BT}\int d^3u\, u^2 \exp\left(-\frac{mu^2}{2k_BT}\right)\right]$$

$$\left(\because \int d^3u\, u^2 \exp\left(-\frac{mu^2}{2k_BT}\right) = \frac{\sqrt{\pi}}{4}\left(\frac{2k_BT}{m}\right)^{\frac{3}{2}}\right.$$
$$\left.\int d^3u\, u^4 \exp\left(-\frac{mu^2}{2k_BT}\right) = \frac{3\sqrt{\pi}}{8}\left(\frac{2k_BT}{m}\right)^{\frac{5}{2}}\right)$$

$$= 4\pi n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \left[\log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \cdot \frac{\sqrt{\pi}}{4}\left(\frac{2k_BT}{m}\right)^{\frac{3}{2}}\right.$$
$$\left. - \frac{m}{2k_BT}\cdot\frac{3\sqrt{\pi}}{8}\left(\frac{2k_BT}{m}\right)^{\frac{5}{2}}\right]$$

$$= n\left[4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \cdot \frac{\sqrt{\pi}}{4}\left(\frac{2k_BT}{m}\right)^{\frac{3}{2}} \log n\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}}\right.$$
$$\left. - 4\pi\left(\frac{m}{2\pi k_BT}\right)^{\frac{3}{2}} \cdot \frac{3\sqrt{\pi}}{8}\left(\frac{2k_BT}{m}\right)^{\frac{3}{2}}\right]$$

$$= n\left[\log n + \frac{3}{2}\log\left(\frac{m}{2\pi k_BT}\right) - \frac{3}{2}\right]$$

The distribution function of an ideal monatomic gas is

$$Z = \frac{1}{N!\,h^{3N}}\int\cdots\int \exp\left[-\frac{1}{2mk_BT}(p_1^2+p_2^2+\cdots+p_{3N}^2)\right] dx_1\cdots dx_{3N}\, dp_1\cdots dp_{3N}$$

$$= \frac{1}{N!\,h^{3N}}\left[\iiint dx_1 dx_2 dx_3\right]^N \left[\int \exp\left[-\frac{p_1^2}{2mk_BT}\right]dp_1\right]^{3N}$$

$$= \frac{V^N}{N!\,h^{3N}}\,(2\pi m k_BT)^{\frac{3N}{2}}$$

where V is volume, N is number of partcle.

The Helmholz free energy is given by

$$F = -k_BT \log Z$$

$$= -k_BT\left[\log\left(\frac{V^N}{N!\,h^{3N}}\right) + \frac{3}{2}N\log(2\pi m k_BT)\right].$$

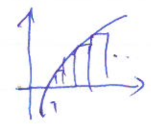
Then, the entropy per unit volume is.

$$S = \frac{1}{V}\left[-\left(\frac{\partial F}{\partial T}\right)_{V,N}\right]$$

$$= \frac{k_B}{V}\left[\log\left(\frac{V^N}{N!\,h^{3N}}\right) + \frac{3}{2}N\log(2\pi m k_BT)\right] + \frac{k_BT}{V}\cdot\frac{3}{2}N\cdot\frac{1}{T}$$

($\because$ Stirling's approximation.

$$\log N! = \log 1 + \log 2 + \cdots + \log N$$



$$\simeq \int_1^N \log x\, dx$$

$$= \left[x(\log x - 1)\right]_1^N$$

$$= N\log N - N \;)$$

$$\left(\because \log\frac{V^N}{N!\,h^{3N}} = N\log V - \log N! - \log h^{3N}\right.$$
$$\simeq N\log V - N\log N + N - \log h^{3N}$$
$$\left.= N\log\frac{V}{N} + N - \log h^{3N}\right)$$

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so that.

$$S = \frac{k_B}{V}\left[N\log\frac{V}{N} + N - \log h^{3N} + \frac{3}{2}N\log(2\pi m k_B T) + \frac{3}{2}N \right]$$

$$= \frac{N}{V}k_B\left[-\log\frac{N}{V} - \log h^3 + \frac{3}{2}\log(2\pi m k_B T) + \frac{5}{2} \right]$$

$$= n k_B \log\left[\frac{1}{n h^3}(2\pi m k_B T)^{\frac{3}{2}} e^{\frac{5}{2}} \right]$$

also.

$$S = -n k_B\left[\log n + \frac{3}{2}\log\left(\frac{1}{2\pi m k_B T}\right) - \frac{3}{2} \right] - n k_B \log h^3 + n k_B$$

$$= -k_B \cdot n\left[\log n + \frac{3}{2}\log\left(\frac{m}{2\pi k_B T}\right) - \frac{3}{2} \right] - n k_B \log h^3 + n k_B + 3 n k_B \log m$$

$$= -k_B H + \text{constant}.$$

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