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COEXISTENCE OF CYCLES OF A CONTINUOUS MAP OF THE LINE INTO ITSELF*†

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The basic result of this investigation may be formulated as follows. Consider the set of natural numbers in which the following relationship is introduced: n_1 precedes n_2 ($n_1 \preceq n_2$) if for any continuous mapping of the real line into itself the existence of a cycle of order n_2 follows from the existence of a cycle of order n_1. The following theorem holds.

Theorem. The introduced relationship transforms the set of natural numbers into an ordered set, ordered in the following way:

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1\,.$$

Every continuous function of a real variable $f(x)$, $-\infty < x < \infty$, generates a continuous map T of the line into itself: $x \mapsto f(x)$. The properties of the map T are basically determined by the structure of the set of its fixed points.

Recall that a point α is called a *fixed point of order k* of the map T if $T^k\alpha = \alpha$ and $T^j\alpha \neq \alpha$ for $1 \leq j < k$. The points $T\alpha$, $T^2\alpha, \ldots, T^{k-1}\alpha$ are also fixed points of order k, and together with α they form a *cycle of order k*.

In this paper we study the problem of the dependence between the existence of cycles of various orders.

The main result of this paper may be stated as follows. Consider the set of natural numbers, in which the following relation has been introduced: n_1 precedes n_2 ($n_1 \preceq n_2$) if for every continuous map of the line into itself the existence of a cycle of order n_1 implies the existence of a cycle of order n_2. This relation is clearly reflexive and transitive and, consequently, the set of natural numbers with this relation is a quasi-ordered set.[1] We prove the following result.

Theorem. *This relation turns the set of natural numbers into an ordered set, which is ordered in the following way:*

$$3 \prec 5 \prec 7 \prec 9 \prec 11 \prec \cdots \prec 3 \cdot 2 \prec 5 \cdot 2 \prec \cdots \prec 3 \cdot 2^2 \prec 5 \cdot 2^2 \prec \cdots \prec$$

$$\prec 2^3 \prec 2^2 \prec 2 \prec 1\,. \qquad (*)$$

*Translated by J. Tolosa.
†The format of the original paper has been retained for historical reasons.
[1]G. Birkhoff, *Lattice Theory*, Amer. Math. Soc., New York, 1948.
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The terminology of ordered sets will not be used in the sequel. The proofs of the theorems actually rely only on Bolzano–Cauchy's intermediate value theorem.

The continuity of T immediately implies that *if the map T has a cycle of order $k > 1$ then it also has a fixed point of first order.*

Theorem 1. *If the map T has a cycle of order $k > 2$ then it also has a cycle of second order.*[2]

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the points of the cycle, with $T\alpha_i = \alpha_{i+1}$, $i = 1, 2, \ldots, k-1$, $T\alpha_k = \alpha_1$. Assume that $\alpha_1 < \alpha_i$ $(i \neq 1)$ and $\alpha_r > \alpha_i$ $(i \neq r)$. Consider the interval (α_1, α_{r-1}) (we assume that $r > 2$; if $r = 2$ one must take the interval (α_k, α_r)). According to whether (α_1, α_{r-1}) contains fixed points of first order or not, we denote by β either the fixed point of first order closest to α_{r-1} or the point α_1 (if (α_1, α_{r-1}) contains fixed points of first order, the nearest point to α_{r-1} exists by the continuity of T). Since $T\alpha_{r-1} = \alpha_r > \alpha_{r-1}$, then $Tx > x$ for $x \in (\beta, \alpha_{r-1}]$. If β is a fixed point of first order then, as can be easily seen, for every integer $j > 0$ there is a neighborhood of β such that $T^j x > x$ for every $x > \beta$ in this neighborhood. If $\beta = \alpha_1$ then $T^j\beta = \alpha_{j+1} > \alpha_1 = \beta$ for $0 < j < k$. On the other hand, $T^{k-r+2}\alpha_{r-1} = \alpha_1 < \alpha_{r-1}$. Therefore, by the continuity of T, there is a point γ on (β, α_{r-1}) such that $T^{k-r+2}\gamma = \gamma$. Since $T\gamma \neq \gamma$, then γ is a fixed point of order l, where $1 < l \leq k - r + 2 < k$. And since there is always a fixed point of order smaller than k, but bigger than one, then there is always a fixed point of second order.

The statements and proofs of the subsequent assertions will be preceded by the following rather trivial lemmas, whose proof is given only for the sake of completeness.

Lemma 1. *If $T^p\alpha = \alpha$ and α is a fixed point of order k of the map T, then p is a multiple of k.*

Indeed, if α is a fixed point of order k then $T^k\alpha = \alpha$ and $T^j\alpha \neq \alpha$ for $j < k$. Let $p = ks + r$, $r < k$. If we assume that $r \neq 0$ then $T^r\alpha \neq \alpha$ and $T^p\alpha = T^r \underbrace{T^k \cdots T^k}_{s \text{ times}} \alpha \neq \alpha$.

Lemma 2. *If T has a fixed point α of order $k = 2^n l$, with l odd, then for the map $S = T^{2^m}$ the point α is a fixed point of order*

$$q = \begin{cases} 2^{n-m} l, & \text{if } n \geq m, \\ l, & \text{if } n \leq m. \end{cases}$$

Proof. By Lemma 1, $T^p\alpha = \alpha$ only for $p = ki$, $i = 1, 2, \ldots$ Assuming that α is a fixed point of S, let us find its order q. We have $S^q\alpha = \alpha$ and $S^j\alpha \neq \alpha$ for $1 \leq j < q$. Since $S^q = T^{2^m q}$ then $S^q\alpha = \alpha$ if and only if $2^m q = ki$, where i is a natural number. Hence, $q = \frac{k}{2^m} i$. The least value of i for which the right-hand side is an integer corresponds to the desired value of q. Indeed, for this q we have, as one can easily see, $S^q\alpha = \alpha$ and $S^j\alpha \neq \alpha$ when $j < q$.

If $k = 2^n l$, with l odd, then $q = 2^{n-m} l i$. For $n \geq m$ we have $i = 1$ and, therefore, $q = 2^{n-m} l$. For $n < m$ we have $i = 2^{m-n}$, i.e. $q = l$.

Corollary. *Under the assumptions of Lemma 2, if $l > 1$ then the fixed point α of the map S has order higher than two.*

Lemma 3. *A point α is a fixed point of order 2^m of the map T if and only if $T^{2^m}\alpha = \alpha$ and $T^{2^{m-1}}\alpha \neq \alpha$.*

[2] This assertion is in Sharkovskiĭ [1960]. Here we give a more accurate proof.

The condition is clearly necessary; let us show it is sufficient. If $T^{2^m}\alpha = \alpha$ then α may be a fixed point of order 2^j, $j = 0, 1, \dots, m$ (Lemma 1). Since $T^{2^{m-1}}\alpha \neq \alpha$ then we also have $T^{2^j}\alpha \neq \alpha$ for every $j < m-1$, since $T^{2^{m-1}} = \underbrace{T^{2^j}(T^{2^j}(\cdots T^{2^j})\cdots)}_{2^{m-j-1}\text{ times}}$.

Thus, α is a fixed point of order 2^m.

Theorem 2. *If the map T has a cycle of order 2^n, $n > 1$, then T has cycles of order 2^i for every $i = 1, 2, \dots, n-1$.*[3]

Let α be a fixed point of order 2^n. We show that T has a fixed point of order 2^m for $1 \leq m < n$.

Set $T^{2^{m-1}} = S$. By Lemma 2, α is a fixed point for S of order 2^{n-m+1}, i.e., of order higher than two. By Theorem 1, S has a fixed point β of second order: $S^2\beta = \beta$ and $S\beta \neq \beta$. Consequently, $T^{2^m}\beta = \beta$ and $T^{2^{m-1}}\beta \neq \beta$.

The following theorem is proved analogously.

Theorem 3. *If the map T has a cycle of order k and k is not a power of two then T has cycles of orders 2^i for $i = 1, 2, 3, \dots$*

Let α be a fixed point of order k. We show that T has a fixed point β of order 2^m, where $m \geq 1$.

Set $T^{2^{m-1}} = S$. By the corollary to Lemma 2, α is a fixed point of order higher than two for S. By Theorem 1, S has a fixed point β of second order. Thus, $S^2\beta = \beta$ and $S\beta \neq \beta$, i.e., $T^{2^m}\beta = \beta$ and $T^{2^{m-1}}\beta \neq \beta$.

From Theorem 3 it follows that there are maps having cycles of arbitrarily high order, since it is always easy to construct a map having a cycle of a prescribed order — in particular, an order different from a power of two.

Theorem 3 also shows that it suffices to specify the function $f(x)$, defining the map T, at finitely many points (forming a cycle), for example, at three points, and there will exist infinitely many cycles, independently of the way we (continuously) change the values of $f(x)$ at the remaining points of the line.

Let us consider the set of fixed points in one cycle. Assume that the points α_1, $\alpha_2 = T\alpha_1, \dots, \alpha_k = T\alpha_{k-1}$ form a cycle of order k. Let us divide the points of the cycle into two sets M_1 and M_2 so that $\alpha_i \in M_1$ if $\alpha_i < T\alpha_i$ and $\alpha_i \in M_2$ if $\alpha_i > T\alpha_i$. Let $\alpha^{M_1} = \max_{\alpha_i \in M_1} \alpha_i$ and $\alpha_{M_2} = \min_{\alpha_i \in M_2} \alpha_i$. We have two possibilities: either $\alpha^{M_1} < \alpha_{M_2}$ or $\alpha^{M_1} > \alpha_{M_2}$.

Lemma 4. *If $\alpha^{M_1} > \alpha_{M_2}$ then the map T has cycles of any order.*

Among all the points belonging to M_1 and bigger than α_{M_2} let us chose one at which the value of the function $f(x)$ defining the map T is the largest. Denote it by β. Since $T\alpha_{M_2} < \alpha_{M_2}$ and $T\beta > \beta$, the set of all fixed points of first order on the interval (α_{M_2}, β) is nonempty and closed (by the continuity of T). Let γ be the greatest fixed point of first order on this interval. Then $T\gamma = \gamma$ and $Tx > x$ on $(\gamma, \beta]$. The interval $(\gamma, T\beta]$ has been chosen so that it contains fixed points of the given cycle (for example, β and $T\beta$). Since applying T repeatedly to any point of the cycle one must close the cycle, then $(\gamma, T\beta]$ must contain at least one point δ of the cycle such that either $T\delta > T\beta$ or $T\delta < \gamma$. The first inequality is impossible. Indeed, if $\delta \in M_2$ then $T\delta < \delta < T\beta$, and if $\delta \in M_1$ then $T\delta < T\beta$ by our choice of β. Thus, on $(\gamma, T\beta]$ there is a point δ of the cycle for which $T\delta < \gamma$ (it is possible that $\delta = T\beta$). Since $Tx > x$ on $(\gamma, \beta]$ then $\delta \in (\beta, T\beta]$. The scheme thus obtained (Fig. 1): $T\gamma = \gamma$, $Tx > x$ on $(\gamma, \beta]$, $\delta \in (\beta, T\beta]$, and $T\delta < \gamma$ (we shall call it an *L-scheme*) guarantees the existence of cycles of all orders.

[3]The statements of Theorems 2 and 3 are in Sharkovskiĭ [1961].

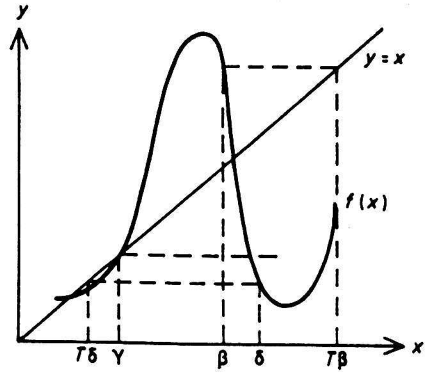

Fig. 1.

Indeed, $T(\gamma, \beta] \supseteq (\gamma, T\beta]$[4] and, consequently, $(\gamma, \beta]$ contains a closed nonempty set of points that is mapped by T, in one step, into the point δ. Denote by δ_1 the smallest of these points. Analogously, since $T(\gamma, \delta_1] = (\gamma, \delta]$, then $(\gamma, \delta_1]$ contains a nonempty closed set of points that are carried by T, in one step, into δ_1. Denote by δ_2 the smallest of these points. Clearly, $\gamma < \delta_2 < \delta_1$ and $T(\gamma, \delta_2] = (\gamma, \delta_1]$. Continuing the process of construction of the points δ_i, we obtain a sequence $\delta > \delta_1 > \delta_2 > \cdots > \delta_{i-1} > \delta_i > \cdots > \gamma$, such that $T\delta_i = \delta_{i-1}$. Evidently, $T^i\delta_{i-1} = T\delta$ and $T^i\delta_i = \delta$. Thus, $T^i\delta_i > \delta_i$ and $T^i\delta_{i-1} < \delta_{i-1}$, so that by the continuity of T^i there is at least one point ρ_i on (δ_i, δ_{i-1}) such that $T^i\rho_i = \rho_i$. Since $T^j(\gamma, \delta_{i-1}] = (\gamma, \delta_{i-j-1}] \subset (\gamma, \delta_1]$ for $j < i - 1$ and $Tx > x$ on $(\gamma, \delta_1]$, then $T^j > x$ on $(\gamma, \delta_{i-1}]$ for $1 \leq j < i$. Hence, $T^j\rho_i \neq \rho_i$ when $1 \leq j < i$, i.e., ρ_i is a fixed point of order i.

This finishes the proof of the lemma.

Remark. If there is a fixed point of first order that is less than α^{M_1} (but greater than $\alpha_{\min} = \min_{i=1,2,\ldots,k} \alpha_i$) then, as before, the map T contains an L-scheme. Hence it follows that, independently of the distribution of the points of the cycle, T has cycles of any order.

If there is a fixed point of first order that is greater than α_{M_2} (but smaller than $\alpha_{\max} = \max_{i=1,2,\ldots,k} \alpha_i$) then T has a scheme representing the reflection of an L-scheme with respect to the point γ as a center. As in the proof of Lemma 4, one shows that this scheme also guarantees the existence of cycles of all orders.

Let us consider the case when $\alpha^{M_1} < \alpha_{M_2}$. We have the following result.

Lemma 5. *If $\alpha^{M_1} < \alpha_{M_2}$ and there is a point $\alpha \in M_1$ such that $T\alpha \in M_1$ as well, then the map T has cycles of odd orders greater than k and cycles of all even orders.*

The lemma also holds for $\alpha \in M_2$ and $T\alpha \in M_2$.

Let us start by singling out a scheme that will lead to the proof of the lemma.

Let $n = \min_{\substack{T^j\xi \leq \alpha \\ \xi \in M_1,\, \xi \geq T\alpha}} j$ and let β be the ξ at which this minimum is attained (or one of them, if there are several such points). Thus, $T^n\beta = \gamma \leq \alpha$ and $T^i\beta > \alpha$ for $i < n$. Let us consider the sequence $T\beta, T^2\beta, T^3\beta, \ldots$ Let $T^l\beta$ be the first point belonging to M_1. It is easy to see that $T^l\beta < T\alpha$. Indeed, if we had $T^l\beta > T\alpha$ (it is clear that $T^l\beta \neq T\alpha$) then the min j indicated earlier would be less than n. Since $T\beta > \beta \geq T\alpha$ then $T\beta \in M_2$ and, consequently, $l \geq 2$. Denote the point $T^{l-1}\beta$ by δ. Then $\delta \in (\beta, T\beta]$ and $T\delta < T\alpha$. This discussion leads to the picture shown in Fig. 2. We shall call such a scheme, an *M-scheme.*

Consider the interval $[\beta, \delta]$ (Fig. 3). Let η be the greatest of the points x of this interval for which $Tx = T\beta$ (it is possible that $\eta = \beta$). On (η, δ) we have $Tx < T\beta$. Since $T\eta = T\beta \geq \delta$ and $T\delta < T\alpha \leq \eta$ then on (η, δ) there is at least one point ζ such that

[4]By $T(\gamma, \beta]$ we mean the set of images of points belonging to $(\gamma, \beta]$.

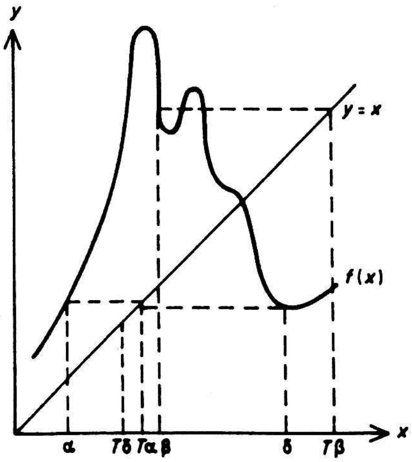

Fig. 2.

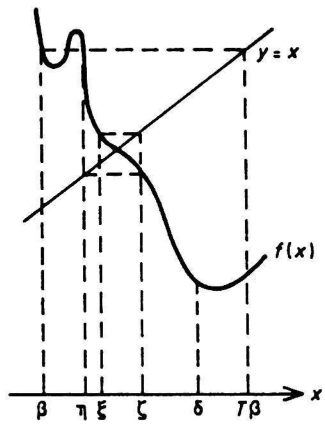

Fig. 3.

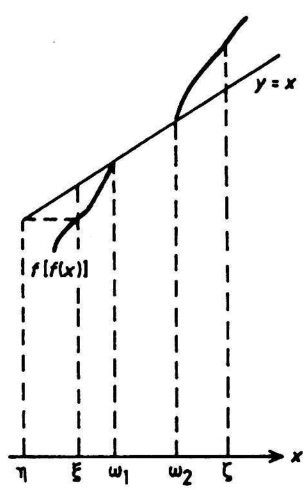

Fig. 4.

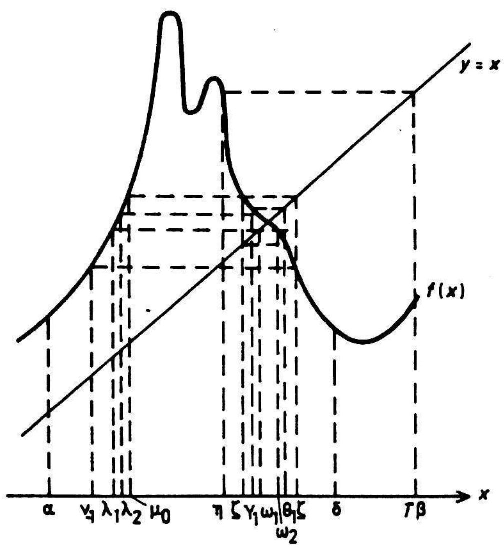

Fig. 5.

$T\zeta = \eta$. If there are several such points on $[\eta, \delta]$, we shall assume that ζ is the smallest of them. Thus, $T\eta = T\beta$, $T\zeta = \eta$, and $\eta < Tx < T\beta$ for every $x \in (\eta, \zeta)$. Further, let us choose the point ξ in $[\eta, \zeta]$ that is the largest of all x for which $Tx = \zeta$. For every $x \in (\xi, \zeta)$ we have $\eta < Tx < \zeta$.

In order to better illustrate the subsequent arguments, let us construct an approximate graph of the function $f(f(x))$ on $[\xi, \zeta]$ (Fig. 4).

We have $T^2\xi = \eta < \xi$, $T^2\zeta = T\beta > \zeta$, and $\eta < T^2x < T\beta$ on (ξ, ζ). Let $\omega_1 \leq \omega_2$ be respectively the smallest and the largest of the points of $[\xi, \zeta]$ for which $T^2x = x$. Clearly, $T\omega_1 = \omega_2$ and $T\omega_2 = \omega_1$, that is, ω_1 and ω_2 form a cycle of order two, or, if $\omega_1 = \omega_2 = \omega$, then ω is a fixed point of first order.[5] Moreover, $T(\xi, \omega_1) = (\omega_2, \zeta)$ and $T(\omega_2, \xi) = (\eta, \omega_1)$.

As we did earlier for the L-scheme, let us construct a sequence $\zeta = \theta_0 > \theta_1 > \theta_2 > \cdots > \omega_2$ such that $T^2\theta_i = \theta_{i-1}$, $T^2(\omega_2, \theta_i) = (\omega_2, \theta_{i-1})$, and a sequence $\xi = \kappa_0 < \kappa_1 < \kappa_2 < \cdots < \omega_1$ such that $T^2\kappa_i = \kappa_{i-1}$ and $T^2(\kappa_i, \omega_1) = (\kappa_{i-1}, \omega_1)$. Consequently, $T^{2i+1}(\omega_2, \theta_i) = (\eta, \omega_1)$ and $T^{2i+2}(\kappa_i, \omega_1) = (\eta, \omega_1)$.

[5]Indeed, $T\omega_2 \geq \omega_1$, since $T^2(T\omega_2) = T\omega_2$ and, hence, $T\omega_2 \notin (\xi, \omega_1)$. Analogously, $T\omega_1 \leq \omega_2$. Therefore $T[\xi, \omega_1] \supseteq [\omega_2, \zeta]$ and on $[\xi, \omega_1]$ there is a point χ such that $T\chi = \omega_2$. For every $x \in [\xi, \omega_1]$ we have $T^2x \leq x$. Thus, $\chi \geq T^2\chi = T\omega_2 \geq \chi$, whence $T^2\chi = \chi$, i.e., $\chi = \omega_1$.

Since $T\alpha < \eta$ and $T\eta = T\beta > \zeta$, on the interval (α, η) there are points at which the value of $f(x)$ is equal to ω_1, ω_2, η, θ_i, and κ_i ($i = 0, 1, 2, \ldots$). We can always find points $\lambda_1, \lambda_2, \mu_0, \nu_{-1} \in (\alpha, \eta)$ such that $T\lambda_1 = \omega_1$, $T\lambda_2 = \omega_2$, $T\mu_0 = \theta_0 = \zeta$, $T\nu_{-1} = \eta$, and $T(\nu_{-1}, \lambda_1) = (\eta, \omega_1)$, $T(\lambda_2, \omega_0) = (\omega_2, \zeta)$. Further, we can find points μ_i, $i = 1, 2, \ldots$ such that $T\mu_i = \theta_i$, $T(\lambda_2, \mu_i) = (\omega_2, \theta_i)$, and points ν_i, $i = 0, 1, 2, \ldots$ such that $T\nu_i = \kappa_i$, $T(\nu_i, \lambda_1) = (\kappa_i, \omega_1)$. Clearly, $T^{2i+2}\mu_i = \eta$, $T^{2i+2}(\lambda_2, \mu_i) = (\eta, \omega_1)$, and $T^{2i+3}\nu_i = \eta$, $T^{2i+3}(\nu_i, \lambda_1) = (\eta, \omega_1)$.

Since $T\eta = T\beta$, we have $T^n\eta = \gamma$ (n is the least positive integer such that $T^n\beta \leq \alpha$). To pass from one point to the cycle to another we need no more than $k - 1$ steps, and therefore $n \leq k - 1$. It is not hard to see that if $\gamma = \alpha$ and $\beta = T\alpha$ then $n = k - 1$.

Let us show that the map T has fixed points of odd order greater than k. Let n be even. In this case, $n + 2i + 3$ ($i \geq 0$) is odd and there is a fixed point of order $s = n + 2i + 3$. Indeed, $T^s\lambda_1 = \omega_1 > \lambda_1$, $T^s\nu_i = \gamma < \nu_i$ and, consequently, on (ν_i, λ_1) there are points x such that $T^s x = x$. Let ρ_s be the largest of these points. We claim that ρ_s is a fixed point of order s. Since s is odd, ρ_s can only be a fixed point of odd order (Lemma 1). Assume that ρ_s is a fixed point of order r, where $r < s$ is odd. We have $T\rho_s \in (\kappa_i, \omega_1)$, and there is a point $\pi' \in (\kappa_{i+\frac{s-r}{2}}, \omega_1)$ such that $T^{s-r}\pi' = T\rho_s$. Since we have $T^2 x < x$ on (κ_j, ω_1), $j = 0, 1, 2, 3, \ldots$ and $T^{s-r} = \underbrace{T^2(T^2(\cdots T^2)\cdots)}_{\frac{s-r}{2}\text{ times}}$, then $T\rho_s < \pi' < \omega_1$. There is a point π'' such that $\rho_s < \pi'' < \lambda_1$ and $T\pi'' = \pi'$. Thus, $\rho_s < \pi'' < \lambda_1$ and $T^s\pi'' = T^{r-1}T^{s-r}T\pi'' = T^{r-1}T^{s-r}\pi' = T^{r-1}T\rho_s = T^r\rho_s = \rho_s < \pi''$ and, therefore, on (π'', λ_1) there is a point ρ'_s at which $T^s\rho'_s = \rho'_s$; but $\rho_s < \rho'_s$, which contradicts the fact that ρ_s is the largest of the points $x \in (\nu_i, \lambda_1)$ such that $T^s x = x$. The odd number $s = n + 3$ ($i = 0$) is never bigger than the smallest odd number bigger than k and, therefore, for n even, we have proved the existence of fixed points of odd order bigger than k.

If n were odd, one would have to use the sequence of points μ_i instead of the sequence $\{\nu_i\}$.

Now we prove that the map T has fixed points of arbitrary even order. Let n be even. In this case one must use the sequence $\{\mu_i\}$. Set $s = n + 2i + 2$; then $T^s\lambda_2 = \omega_2 > \lambda_2$, $T^s\mu_i = \gamma < \mu_i$ and, therefore, on (λ_2, μ_i) there are points x such that $T^s x = x$. Let σ_s be one such point. We claim that for $s \geq 2k - 2$, σ_s is a fixed point of order s. Indeed, since $T^s\sigma_s = \sigma_s$, then σ_s is either a fixed point of order s or a fixed point of smaller order r, and s is a multiple of r (Lemma 1). Clearly, $r \leq \frac{s}{2}$ and hence if $T^j\sigma_s \neq \sigma_s$ for $1 \leq j \leq \frac{s}{2}$ then σ_s is a fixed point of order s. On (λ_2, μ_i) we have $T^j x > \eta > x$ for every $1 \leq j \leq s - n$, since $T^j(\lambda_2, \mu_i) \subset (\eta, \zeta)$ for $j < s - n$. Thus, σ_s is a fixed point of order s for $s - n \geq \frac{s}{2}$; and this inequality is always true for $s \geq 2k - 2$.

When n is odd, the existence of fixed points of even order $s \geq 2k - 2$ is proved analogously, only now using the points ν_i.

It remains to prove that T has fixed points of even order smaller than $2k - 2$. Before completing the proof of Lemma 5, let us prove the following result.

Lemma 6. *If the map T has a cycle of odd order then it has cycles of any even order.*

Consider the sets M_1 and M_2. If $\alpha^{M_1} > \alpha_{M_2}$ then there are cycles of all orders (Lemma 4). Assume that $\alpha^{M_1} < \alpha_{M_2}$. The points of a cycle of odd order k will also form a cycle of order k for the map $S = T^2$ (Lemma 2). For the map S one can construct sets M_1^2 and M_2^2, as we did with the sets M_1 and M_2, considering that a point α_i is in M_1^2 if $\alpha_i < T^2\alpha_i$ and $\alpha_i \in M_2^2$ if $\alpha_i > T^2\alpha_i$. Let $\alpha^{M_1^2}$ be the largest point of M_1^2 and let $\alpha_{M_2^2}$ be the smallest point in M_2^2. Let us prove that S has cycles of all orders.

Since $\alpha^{M_1} < \alpha_{M_2}$, the map T has a fixed point γ of first order such that $\alpha^{M_1} < \gamma < \alpha_{M_2}$. This is a fixed point of first order also for $S = T^2$. If $\alpha^{M_1^2} \neq \alpha^{M_1}$ (and, consequently, also

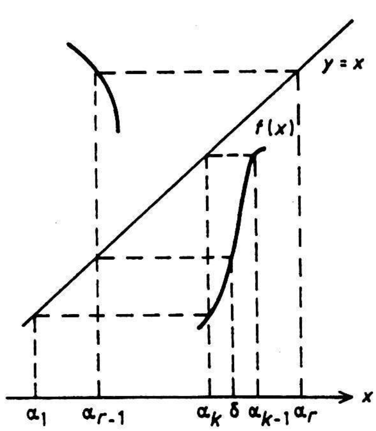

Fig. 6.

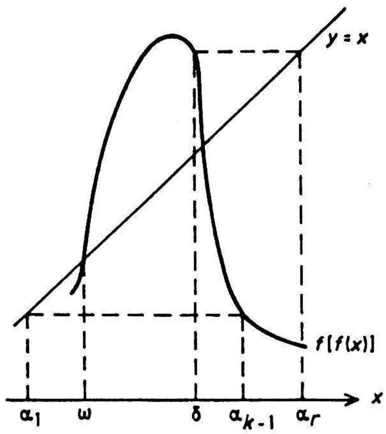

Fig. 7.

$\alpha_{M_2^2} \neq \alpha_{M_2}$) then either $\alpha^{M_1^2} \in M_2$ and $\gamma < \alpha^{M_1^2}$ or $\alpha_{M_2^2} \in M_1$ and $\gamma > \alpha_{M_2^2}$. It remains to use the remark following Lemma 4.

Assume that $\alpha^{M_1^2} = \alpha^{M_1}$ and hence also $\alpha_{M_2^2} = \alpha_{M_2}$, $M_1^2 = M_1$, and $M_2^2 = M_2$. Let α_1 be the smallest of all the α_i, $i = 1, 2, \ldots, k$. Then $\alpha_k \in M_2$. Since $\alpha_{k-1} > \alpha_1$ then $\alpha_{k-1} \in M_2^2$ and, consequently, $\alpha_{k-1} \in M_2$. Thus, $\alpha_{k-1} > \alpha_k$. Let α_r be the largest of all points α_i, $i = 1, 2, \ldots, k$. Then $\alpha_{r-1} \in M_1$ and hence $\alpha_1 < \alpha_{r-1} < \alpha_k$. Since on (α_k, α_{k-1}) the function $f(x)$ takes at least all the values of the interval (α_1, α_k), then on (α_k, α_{k-1}) there is a point δ such that $T\delta = \alpha_{r-1}$. Finally, let ω be the largest of the points $x \in [\gamma, \delta)$ for which $Sx = x$ (on $[\gamma, \delta)$ there is at least one point x such that $Sx = x$, since $S\gamma = \gamma$).

Thus, we have: $S\omega = \omega$, $S\delta = \alpha_r > \delta$, $Sx > x$ on $(\omega, \delta]$, $\alpha_{k-1} \in (\delta, \alpha_r)$, and $S\alpha_{k-1} = \alpha_1 < \omega$. We have singled out an L-scheme for the map S (see the proof of Lemma 4), which guarantees the existence of cycles of all orders for S.

The fact that S has cycles of all orders immediately implies the existence of cycles of even order for T. For example, let us prove that T has a cycle of order $l = 2l_1$.

Let α be a fixed point of order l_1 of S. This means that $S^{l_1}\alpha = \alpha$ and $S^{j_1}\alpha \neq \alpha$ for $1 \leq j_1 < l_1$, i.e., $T^l\alpha = \alpha$ and $T^j\alpha \neq \alpha$, where j is any even number less than l. Since $S\alpha \neq \alpha$, then also $T\alpha \neq \alpha$. Hence, either α is a fixed point of order l for T or α is a fixed point of odd order l_2 (but not one), and $l_2 \leq l_1$.[6] But to a cycle of odd order we can always apply either Lemma 4 or Lemma 5. Indeed, since the cycle contains an odd number of points, then there are more points either in M_1 or in M_2. To fix the ideas, assume that there are more points in M_1 than in M_2. Then necessarily there is a point μ in M_1 such that $T\mu \in M_1$, since otherwise the number of points of M_1 could not be bigger than that of M_2. Thus, since the map T, having a cycle of order l_2, satisfies the assumptions of either Lemma 4 or Lemma 5, then T must have cycles of even order $\geq 2l_2 - 2$, and hence, of order l. Lemma 6 is proved.

This concludes the proof of Lemma 5. Since in its proof we have already established the existence of cycles of odd order (bigger than k), it follows that there are cycles of all even orders as well.

All these arguments imply the following result.

Theorem 4. *If the map T has a cycle of odd order k then it has cycles of all odd orders bigger than k and all even orders.*

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[6]Generally speaking, Lemma 2 implies that for T the point α is a fixed point of order $2l_1$, if l_1 is even, and either $2l_1$ or l_1, if l_1 is odd.

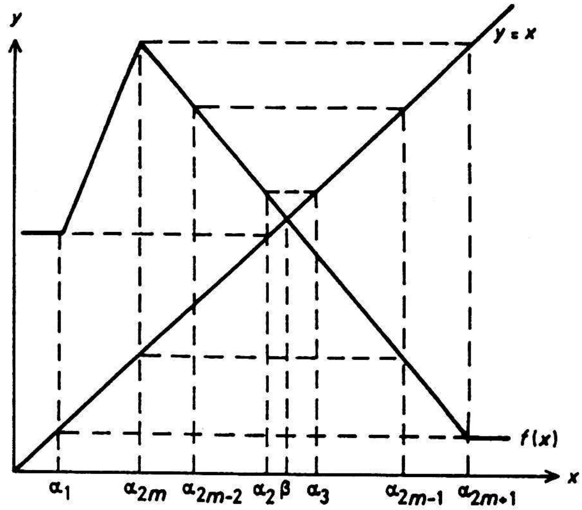

Fig. 8.

Theorem 4 cannot be sharpened. Now we will construct an example of a map T having a cycle of order $2m+1$ but having no cycles of order $2j-1$ for $j = 2, 3, \ldots, m$.

Assume that the points α_i, $i = 1, 2, \ldots, 2m+1$ form a cycle of order $2m+1$, with $\alpha_{i+1} = T\alpha_i$, $i = 1, 2, \ldots, 2m$, $\alpha_1 = T\alpha_{2m+1}$, and assume that $\alpha_1 < \alpha_{2m} < \alpha_{2m-2} < \cdots < \alpha_2 < \alpha_3 < \cdots < \alpha_{2m+1}$. Assume that the continuous function $f(x)$ defining the map T is equal to α_2 for $x \le \alpha_1$, is equal to α_1 for $x \ge \alpha_{2m+1}$, and for $\alpha_1 \le x \le \alpha_{2m+1}$ is a piecewise-linear function with vertices at the points (α_1, α_2), $(\alpha_2, \alpha_3), \ldots, (\alpha_{2m}, \alpha_{2m+1})$, $(\alpha_{2m+1}, \alpha_1)$ of the x, y-plane.

It is not hard to see that

$$T^{2j-1}(\alpha_1, \alpha_{2m}] = (\alpha_{2j}, \alpha_{2m+1}],$$

$$T^{2j-1}(\alpha_{2i+2}, \alpha_{2i}] = \begin{cases} [\alpha_{2(i+j)-1}, \alpha_{2(i+j)+1}), & \text{if } 2 \le i+j \le m, \\ (\alpha_{2(i+j-m)}, \alpha_{2m+1}], & \text{if } m+1 \le i+j \le 2m-1, \end{cases}$$

$$T^{2j-1}[\alpha_{2i+1}, \alpha_{2i+3}) = \begin{cases} (\alpha_{2(i+j)+2}, \alpha_{2(i+j)}], & \text{if } 2 \le i+j \le m-1, \\ (\alpha_1, \alpha_{2m}], & \text{if } i+j = m, \\ [\alpha_1, \alpha_{2(i+j-m)+1}), & \text{if } m+1 \le i+j \le 2m-1, \end{cases}$$

$$i = 1, 2, \ldots, m-1, \qquad j = 1, 2, \ldots, m.$$

If $T\beta = \beta$ then

$$T^{2j-1}(\alpha_2, \beta) = (\beta, \alpha_{2j+1}), \quad 1 \le j \le m,$$

$$T^{2j-1}(\beta, \alpha_3) = \begin{cases} (\alpha_{2j+2}, \beta), & \text{if } 1 \le j < m, \\ (\alpha_1, \beta), & \text{if } j = m. \end{cases}$$

Finally, observe that $T^{2j-1}x = \alpha_{2j}$ for every $x \le \alpha_1$ and $T^{2j-1}x = \alpha_{2j-1}$ for $x \ge \alpha_{2m+1}$, $1 \le j \le m$.

Thus, $T^{2j-1}x > x$ when $x < \beta$ and $T^{2j-1}x < x$ if $x > \beta$, for every $1 \le j \le m$, and, consequently, the map T has no cycles of order $3, 5, \ldots, 2m-1$.

Theorem 4 can be generalized to the case when T has a cycle of any order that is not a power of two.

Theorem 5. *If the map T has a cycle of order $k = 2^n l$, where $l > 1$ is odd, then T has a cycle of order $2^n r$, where $r > l$ is any odd number, and cycles of order $2^{n+1}s$, where s is any natural number.*

Proof. If $n = 0$ we obtain Theorem 4, which has already been proved. Assume that the theorem is true for $n = m - 1$, and let us then prove that it also holds for $n = m$.

Assume that T has a fixed point α of order $2^m l$. Let us prove, for instance, that in this case T also has a fixed point of order $2^m r_0$, where $r_0 > l$ is odd. The point α is a fixed point of order $2^{m-1}l$ for the map $S = T^2$ (Lemma 2) and, by our assumption, S must have a fixed point β of order $2^{m-1}r_0$. This means that $S^{2^{m-1}r_0}\beta = \beta$ and $S^j\beta \neq \beta$ for $j = 1, 2, 3, \ldots, 2^{m-1}r_0 - 1$, that is, $T^{2^m r_0}\beta = \beta$ and $T^i\beta \neq \beta$ for every even i less than $2^m r_0$. We have $T\beta \neq \beta$, since otherwise we would have $S\beta = \beta$. Thus, either β is a fixed point of order $2^m r_0$ for T, or β is a fixed point of odd order, and then, by Theorem 3, T has fixed points of every even order, and, therefore, there is a fixed point γ of order $2^m r_0$.

The proof that T has also fixed points of order $2^{m+1}s$, where s is any natural number, is completely similar.

Thus, Theorem 5 holds for every n.

Theorems 2, 3, and 5, and the fact that there is always a fixed point of first order if there are fixed points of higher order, can be put together in one single theorem.

Theorem 6. *If the map T has a cycle of order 2^n, $n > 0$, then T also has cycles of order 2^i, $i = 0, 1, \ldots, n-1$. If T has a cycle of order $2^n(2m+1)$, $n \geq 0$, $m > 0$, then it also has cycles of order 2^i, $i = 0, 1, \ldots, n$, and cycles of order $2^n(2r+1)$, $r = m+1, m+2, \ldots$, and of order $2^{n+1}s$, $s = 1, 2, 3, \ldots$*

Remark. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the points of a given cycle of order k, and let $a = \min_i \alpha_i$, $b = \max_i \alpha_i$. Theorem 6 concerns only the points of the interval $[a, b]$. Outside $[a, b]$ the map may not have any points of cycles. So, the points of the cycles of the map $\overline{T}$ defined as $\overline{T}x = Ta$ for $x \leq a$, $\overline{T}x = Tx$ for $a \leq x \leq b$, and $\overline{T}x = Tb$ for $x \geq b$, belong to $[a, b]$.

Let us define the *diameter* of the cycle $\alpha_1, \alpha_2, \ldots, \alpha_k$ as the number $d_{\alpha_1,\alpha_2,\ldots,\alpha_k} = \max_{1 \leq i,j \leq k} |\alpha_i - \alpha_j|$. For every n, following k in (*), there is a cycle $\beta_1, \ldots, \beta_k$ for which $d_{\beta_1,\beta_2,\ldots,\beta_n} < d_{\alpha_1,\alpha_2,\ldots,\alpha_k}$. Moreover, as is easily seen, there is a constant C, depending on $\alpha_1, \alpha_2, \ldots, \alpha_k$, such that for every $m > 1$ following k in (*) there is a cycle $\gamma_1, \ldots, \gamma_m$ for which $d_{\gamma_1,\ldots,\gamma_n} > C$.

Let us construct an example showing that Theorem 6 completely solves the problem on the existence of cycles of some orders depending on the existence of cycles of other orders.

In the x, y-plane let there be given points $A^{(1)}(x^{(1)}, y^{(1)})$, $A^{(2)}(x^{(2)}, y^{(2)}), \ldots,$ $A^{(k)}(x^{(k)}, y^{(k)})$, with $x^{(1)} < x^{(2)} < \cdots < x^{(k)}$. These points define the following continuous function $f(x)$: for $x \in [x^{(1)}, x^{(k)}]$, $f(x)$ is a piecewise-linear function with vertices at the points $A^{(1)}, \ldots, A^{(k)}$; for $x \leq x^{(1)}$, $f(x) = y^{(1)} =$ const., and for $x \geq x^{(k)}$, $f(x) = y^{(k)} =$ const. We shall denote by $T_{A^{(1)}A^{(2)}\ldots A^{(k)}}$ the map given by this function.

We shall carry out the construction without getting into the details.

Let us take in the plane's two points A_1 and A_2, symmetric with respect to the bisector of the first and third quadrants. It is easy to see that the map $T_{A_1A_2}$ has only cycles of first and second orders. Let us draw through A_1 a line a_1 perpendicular to the bisector, and through A_2, a line a_2 parallel to the bisector. Let us take on a_1 points A_{11} and A_{12} symmetric with respect to A_1, and on a_2, points A_{21} and A_{22} symmetric with respect to A_2, and such that $|x_{11} - x_{12}| = |x_{21} - x_{22}| \leq \frac{|x_1 - x_2|}{2}$ (we denote by x_r the x-coordinate of A_r). It can be seen that the map $T_{A_{11}A_{12}A_{21}A_{22}}$ has only cycles of first, second, and fourth orders. Now through the points A_{11}, A_{12} and A_{21} we must draw lines a_{11}, a_{12}, and a_{21}

perpendicular to the bisector, and through A_{22}, a line a_{22} parallel to the bisector (clearly, a_{11} and a_{12} will coincide with a_1, and a_{22}, with a_2). Next, as before, on these lines one must take points $A_{111}, A_{112}, A_{121}, \ldots, A_{222}$ symmetric with respect to $A_{11}, A_{12}, A_{21}, A_{22}$, such that

$$|x_{111} - x_{112}| = |x_{121} - x_{122}| = |x_{211} - x_{212}| = |x_{221} - x_{222}| \leq \frac{|x_{11} - x_{12}|}{2},$$

etc. Observe that one can draw parallel (and perpendicular) lines to the bisector through any points and through any number of these points, provided this number is odd.

The map $T_{A_{\underbrace{11\cdots 1}_{n+1}}A_{\underbrace{11\cdots 12}_{n+1}}\cdots A_{\underbrace{22\cdots 2}_{n+1}}}$ has only cycles of orders $1, 2, 2^2, \ldots, 2^{n+1}$. To fix the ideas, assume that the line $\alpha_{\underbrace{11\cdots 11}_{n+1}}$ is perpendicular to the bisector. We replace the two points $A_{\underbrace{111\cdots 11}_{n+1}}(x_{11\cdots 11}, y_{11\cdots 11})$, $A_{\underbrace{111\cdots 12}_{n+1}}(x_{11\cdots 12}, y_{11\cdots 12})$, in the x, y-plane by the points $A_{10}(x_{10}, y_{10}), A_{20}(x_{20}, y_{20}), \ldots, A_{2m+1,0}(x_{2m+1,0}, y_{2m+1,0})$ (see Fig. 10) where

$$x_{10} = x_{\underbrace{11\cdots 11}_{n+1}} < x_{2m,0} < x_{2m-2,0} < \cdots < x_{20} < x_{30} < \cdots < x_{2m+1,0} = x_{\underbrace{11\cdots 12}_{n+1}},$$

$$y_{i0} = x_{i+1,0} + (y_{\underbrace{11\cdots 12}_{n+1}} - x_{\underbrace{11\cdots 11}_{n+1}}), \quad i = 1, 2, \ldots, 2m, \quad y_{2m+1,0} = y_{\underbrace{11\cdots 12}_{n+1}}.$$

It is not hard to see that the map $T_{A_{10}A_{20}\cdots A_{2m+1,0}A_{\underbrace{11\cdots 21}_{n+1}}A_{\underbrace{11\cdots 22}_{n+1}}\cdots A_{\underbrace{22\cdots 22}_{n+1}}}$ has cycles of orders $1, 2, 2^2, \ldots, 2^n, 2^n(2r+1)$, for $r \geq m$, and $2^{n+1}s$, $s > 0$, and has no cycles of any other orders.

Theorem 6 and this example prove the theorem stated at the beginning of this paper.

The following result is related to Theorems 1–6.

Theorem 7. *Between any two points of a cycle of order $k > 1$ there is at least one point of a cycle of order $l < k$.*

Let $\alpha > \beta$ be points of a cycle of order k, and let n_α, n_β be the number of points of this cycle smaller than α and β, respectively. Clearly, $k > n_\alpha > n_\beta > 0$. There are n_α distinct positive integers s_i, $i = 1, 2, \ldots, n_\alpha$, smaller than k and such that $T^{s_i}\alpha < \alpha$. Since $n_\alpha > n_\beta$, there is an s_{i_0}, $1 \leq i_0 \leq n_\alpha$, such that $T^{s_{i_0}}\alpha < \alpha$ and $T^{s_{i_0}}\beta > \beta$. But this means that there is a point $\gamma \in (\beta, \alpha)$ for which $T^{s_{i_0}}\gamma = \gamma$; γ is a point of a cycle of order $1 \leq s_{i_0} \leq k$.

In conclusion, we observe that all the results may be translated into the language of periodic solutions of the functional equation $y(x+1) = f(y(x))$ (where x runs through a discrete sequence of values). For example, if a map $y \mapsto f(y)$ of the line into itself is continuous, then 1) if the functional equation has a periodic solution of period k then it also has periodic solutions of any period following k in (*), and 2) if the equation has no periodic solution with period k then it has no periodic solutions of any period preceding k in (*).

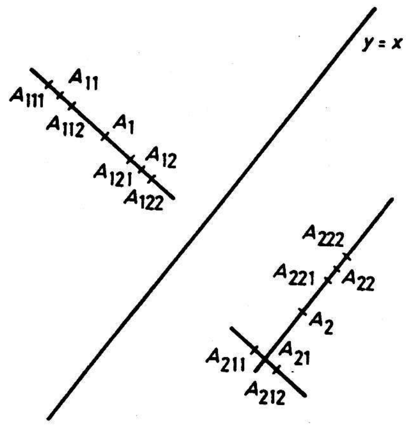

Fig. 9.

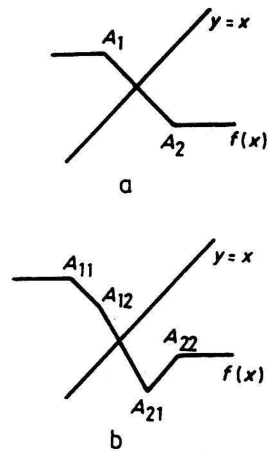

Fig. 10.

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