Exact Expanding Solution of Gas Sphere under Self-gravity

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Abstract:

A new dynamical solution for a gas sphere under self-gravity is presented to describe a development of a gas sphere from a motion-less state to a state of expansion with a constant speed and a reflection phenomenon in the dynamics of the surface of the sphere.

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Basic equations

The equations governing the spherically symmetrical flow of a polytropic gas of adiabatic index γ under the influence of its own gravitation are

$$\rho_t + u\rho_r + \rho\left(u_r + \frac{2}{r}u\right) = 0, \tag{1}$$

$$u_t + uu_r + \rho^{\gamma-2}\rho_r + \sigma_r = 0, \tag{2}$$

$$\sigma_{rr} + \frac{2}{r}\sigma_r = \rho, \tag{3}$$

where the density ρ, radial velocity u, gravitational potential σ and radius r are normalized variables.

The Emden solution

In this model, The equation governing the static equilibrium of a polytropic gas sphere is called the Emden equation.

$$\rho^{\gamma-2}\rho_r + \sigma_r = 0,$$
$$\sigma_{rr} + \frac{2}{r}\sigma_r = \rho,$$

or when $\gamma \neq 1$

$$\frac{d^2\theta}{dr'^2} + \frac{2}{r'}\frac{d\theta}{dr'} + \theta^{\frac{1}{\gamma-1}} = 0, \tag{4}$$

where $\theta \equiv \rho^{\gamma-1}$, $r' = \sqrt{\gamma-1}r$.

The solution of Eqs (4) is called the Emden solution.

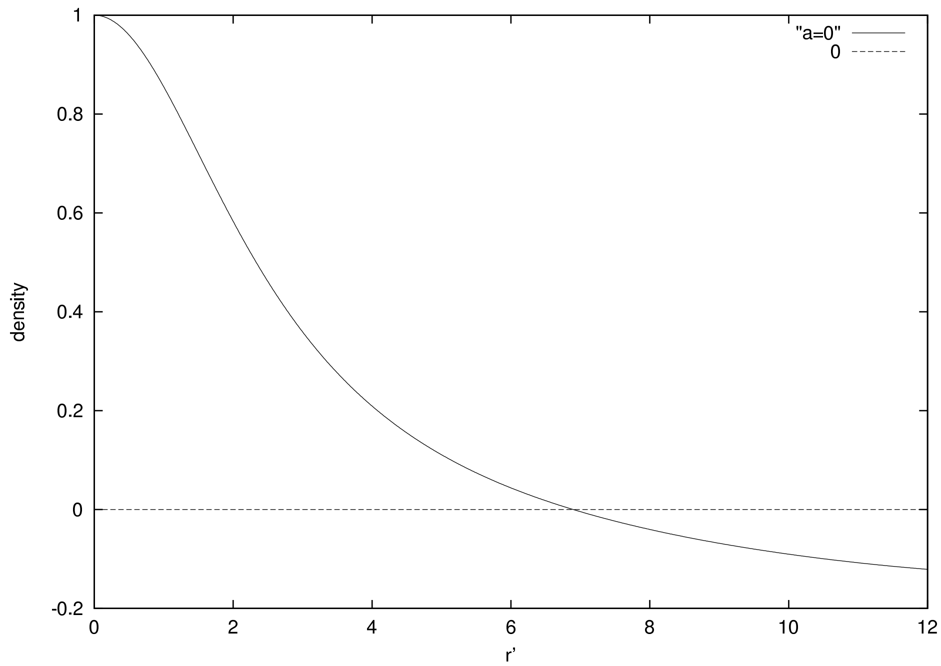

図 1: For $\gamma = 4/3$, Numerical Integration with boundary condition $\theta(0) = 1$, $\frac{d\theta}{dr}(0) = 0$.

Quasi-statical change of stellar structure expressed by the Emden solution is believed to describe the evolution of star when the Emden solution is dynamically stable.

The dynamical stability is not guaranteed unless the polytropic index N is less than 3. A gas sphere of $N = 3$ is borderline to the dynamical stability and is also interesting since the polytropic index 3 corresponding to the adiabatic constant $\gamma = 4/3$.

There are some interesting stellar gases with $\gamma = 4/3$, such as a radiation pressure dominant gas, an extremely relativistic gas and a gas with degenerate electrons.

The purpose of this study is to extend the static Emden solution to dynamical state and is to construct similarity solutions.

Similarity consideration

To seek a similarity transformation, we carry out Lie symmery analysis.

The basic idea of Lie symmetry analysis is to find a Lie group of transformations under wich differential equations are invariant.

we put one parameter infinitesimal group of transformations and the generator

$$
\begin{aligned}
\overline{t} &= t + \epsilon\tau(r, t, \rho, u, \phi), \\
\overline{r} &= r + \epsilon\xi(r, t, \rho, u, \phi), \\
\overline{\rho} &= \rho + \epsilon\psi^{\rho}(r, t, \rho, u, \phi), \\
\overline{u} &= u + \epsilon\psi^{u}(r, t, \rho, u, \phi), \\
\overline{\phi} &= \phi + \epsilon\psi^{\phi}(r, t, \rho, u, \phi),
\end{aligned}
$$

$$\mathbf{V} = \tau\partial_t + \xi\partial_r + \psi^{\rho}\partial_{\rho} + \psi^{u}\partial_u + \psi^{\phi}\partial_{\phi}.$$

The coordinates of this generator are found by the prolongation formula.

The similarity transformation can be obtained by integrating the charastristic equation

$$\frac{dt}{\tau} = \frac{dr}{\xi} = \frac{d\rho}{\psi^{\rho}} = \frac{du}{\psi^{u}} = \frac{d\phi}{\psi^{\phi}}.$$

Eqs.(1), (2) and (3) adomits a scaling symmetry

$$\mathbf{V} \equiv (t+t_0)\partial_t - (\gamma-2)r\partial_r - 2\rho\partial_\rho - (\gamma-1)u\partial_u - \gamma\phi\partial_\phi,$$

where $\phi = \sigma_r$ and t_0 is an arbitrary constant.

In finding a similarity transformation, we introduce a new independent variable T defined as

$$dT = \frac{dt}{t+t_0},$$

and scaled variables

$$\rho = \frac{R(x,T)}{(t+t_0)^2}, \quad u = \frac{U(x,T)}{(t+t_0)^{\gamma-1}}, \quad \phi = \frac{\Phi(x,T)}{(t+t_0)^{\gamma}},$$

where

$$x = \frac{r}{(t+t_0)^{2-\gamma}}, \quad T = \log(t+t_0).$$

Let us make the following ansatz

$$R(x,T) = \tilde{R}(y) \exp\left\{ \int \left(2 - 3\overline{U}\right) dT \right\},$$
$$U(x,T) = \overline{U}(T)x,$$
$$\Phi(x,T) = \tilde{\Phi}(y) \exp\left\{ \int \left(\gamma - 2\overline{U}\right) dT \right\},$$

where

$$y = x \exp\left\{ \int \left(2 - \gamma - \overline{U}\right) dT \right\}.$$

This ansatz ensures that Eq. (1) is satisfied automatically. For $\gamma = 4/3$, Eqs. (2) and (3) lead to

$$ay + \tilde{R}^{\gamma-2}\tilde{R}_y + \tilde{\Phi} = 0, \tag{5}$$

$$\overline{U}_T + \overline{U}(\overline{U} - 1) = a \exp\left\{ \int \left(2 - 3\overline{U}\right) dT \right\}, \tag{6}$$

$$\tilde{\Phi}_y + \frac{n}{y}\tilde{\Phi} = \tilde{R}, \tag{7}$$

where a is an arbitrary constant and we call a as the acceleration parameter.

Dynamical solution of gas sphere

Eqs. (5) and (7) give a modified Emden equation for the density profile $\tilde{R}(y)$ (cf: Eq.(4), Fig2)

$$\theta_{y'y'} + \frac{2}{y'}\theta_{y'} + \theta^3 + 3a = 0, \quad \theta = \tilde{R}^{1/3}, \quad y' = \frac{y}{\sqrt{3}}, \quad (8)$$

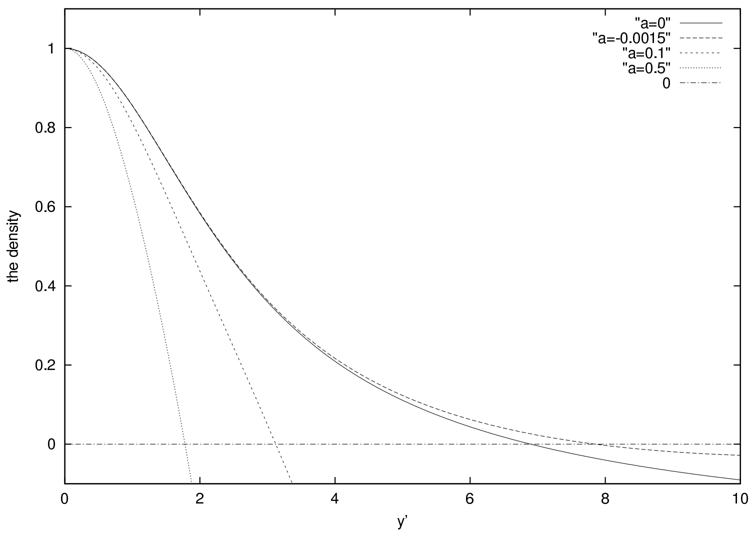

図 2: For $\gamma = 4/3$, Numerical Integration with boundary condition $\theta(0) = 1$, $\frac{d\theta}{dr}(0) = 0$

The density profiles has the first zero at finite $y' = Z$ for $a_0 > -0.00219$.

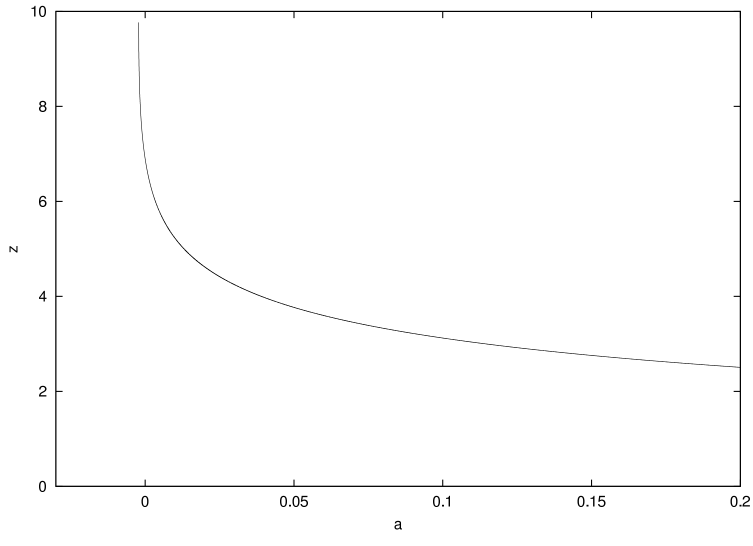

図 3: z vs. the accerelation parameter a

The solution is written as

$$y = \frac{r}{f}, \quad \rho = \frac{\tilde{R}(y)}{f^3}, \quad u = f_t y, \quad \phi = \frac{\tilde{\Phi}(y)}{f^2}, \tag{9}$$

where $\tilde{R}(y)$ is expressed by a localized solution of Eq. (8) and $f(t)$ is given by

$$f(t) = \exp\left(\int^T \overline{U} dT\right)$$

and satisfies the following equation

$$\frac{1}{2} f_t^2 + \frac{a}{f} = b, \tag{10}$$

where b is an arbitrary constant.

For $a = 0$ and $b = 0$, $\rightarrow$ the static Emden solution.

For $a = 0$ and $b > 0$, $\rightarrow$ the expanding Emden solution.

For the other values of a and $b = 0$,
$\rightarrow$**a new class of expanding solution !!**

A new class of solution

To demonstrate the new solution, we look at the velocity at the surface.

Since the distance between the first zero and the center surface is given by $r = f(t)z$, we have

$$\text{the velocity} \quad \tilde{u} = \sqrt{3} f_t z,$$

the accerelation rate $\quad \tilde{u}_t = a\sqrt{3} z / f^2.$

We consider an initial value problem for Eq.(10)

(1) The case in $a > 0$

Since $f \to \infty$ as $t \to \infty$, $f_t \to \sqrt{2b}$ asymptotically.

$$\lim_{t\to\infty} \tilde{u} = \sqrt{6b}z, \quad \lim_{t\to\infty} \tilde{u}_t = 0.$$

In this case, we can take a motionless initial condition so that $f_t|_{t=0} = 0$ by choosing the arbitrary (positive) constant b as $b = a$.

The expansion speed of the surface is depicted for $a = b = 0.01$ in Fig.4.

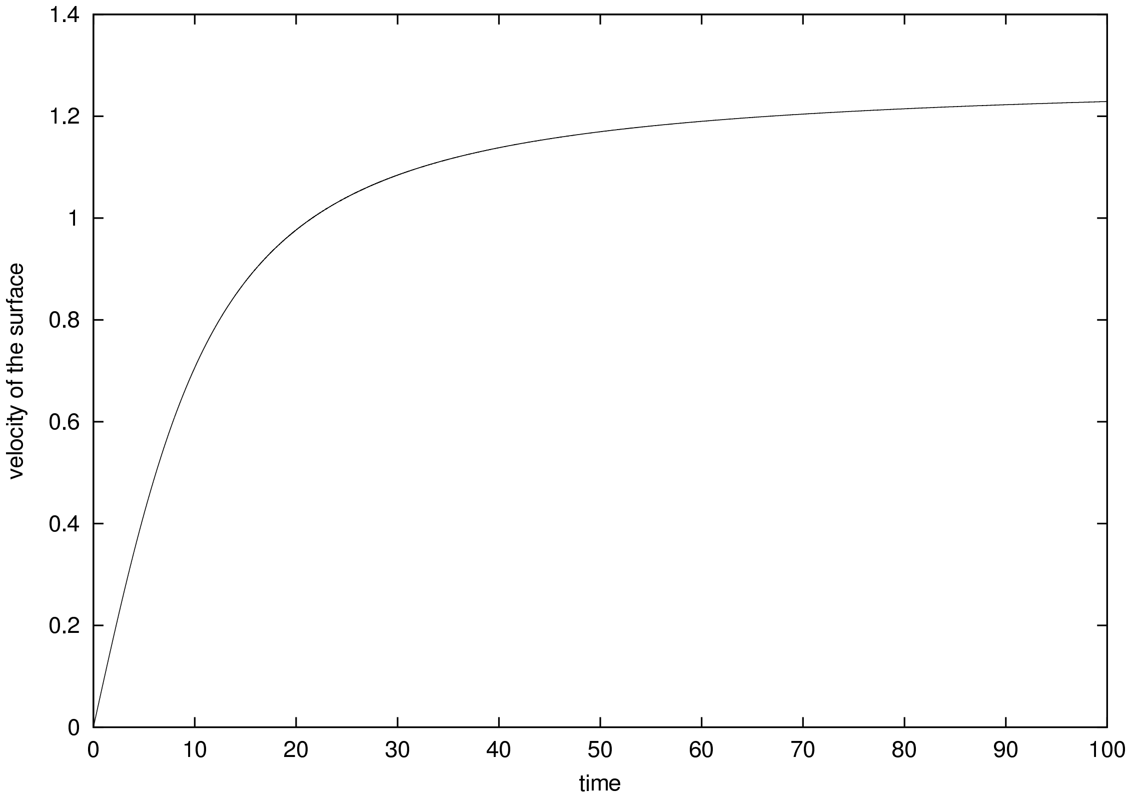

図 4: Expansion velocity of the surface for a=b=0.01.

(2) The case in $a < 0$.

This case is the radially decelerating case, where we take $f_t|_{t=0} > 0$ as an initial condition.

In this case, Eq.(10) reads

$$\frac{1}{2} f_t^2 = b + \frac{|a|}{f}.$$

For $b > 0$, there are no reflection points and the expanding speed of the surface decreases to a constant speed , that is,

$$\lim_{t\to\infty} \tilde{u} = \sqrt{6b} z, \quad \lim_{t\to\infty} \tilde{u}_t = 0.$$

We illustrate the velocity of the surface for $a = -0.002$ and $b = 0.001$ in Fig. 5.

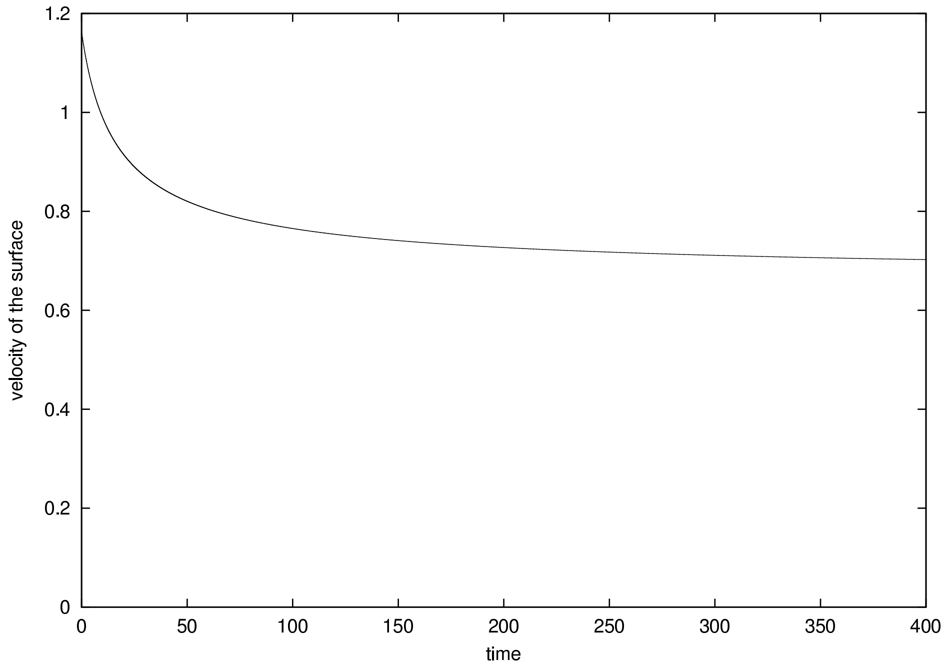

図 5: Velocity of the surface for a=-0.002 and b=0.001.

For $b < 0$, a reflection point appears at $f = a/b$. The deceleration of the surface stops at the reflection point and then the gas sphere began to shrinks.

The reflection time t_r is given as

$$t_r = \left| \frac{\sqrt{(2b - 2a)}}{2b} - \frac{a}{b\sqrt{-2b}} \arctan \sqrt{\frac{2b - 2a}{-2b}} \right|.$$

In this reflection case, the velocity of the surface is depicted for $a = -0.002$ and $b = -0.001$ in Fig.6, where $t_r = 57.485$.

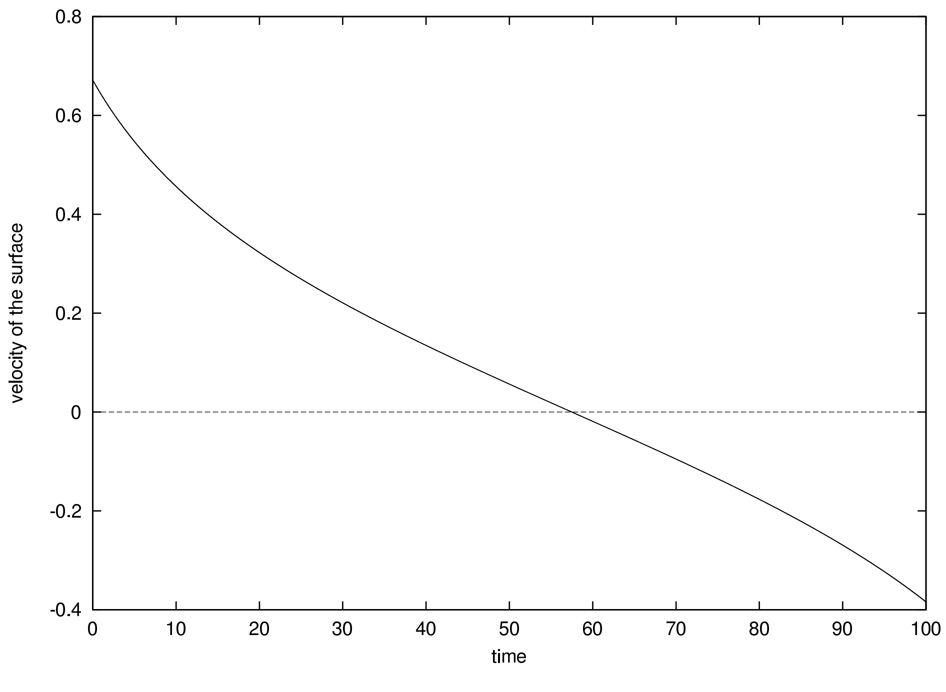

図 6: Velocity of the surface for a=-0.002 and b=-0.001.

Summary

We present a new dynamical solution for a gas sphere under self-gravity, which not only unifies the three Emden-type solutions for the polytropic index $N = 3$ but also describes the following interesting phenomena.

For positive values of acceleration parameter a, the radius of the gas sphere decreases from the Emden's radius due to inward inertia force, while the gas sphere expands for the negative a and the distinct surface disappears when $a < -0.00219$.

The new solution also describes acceleration of the surface of a gas sphere from a motion-less state to a state of expansion with a constant speed for the positive a and a reflection phenomenon in the dynamics of the surface of a sphere for the negative a.