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Basic Seminar IIA

Section 6.1 - 6.4: Gas Dynamics

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Abstract

- **Thermal properties of a perfect gas**
  - EoS, Internal Energy, Entropy, Enthalpy, Bernoulli's Principle

- **Acoustic waves**
  - Small Perturbation, Linear Perturbation Technique, Adiabatic

- **Emission of acoustic waves**
  - Acoustic Radiation, Quadratic Term

- **Steepening into shock waves & Method of characteristic**
  - Large Amplitude, Characteristic Curve

6.1 Thermal properties of a perfect gas

- **Equation of state (EoS) for perfect gas**

$$\rho = mn$$
$$p = n\kappa_B T$$

→

$$p = R\rho T \quad \textbf{(6.1)} \qquad R = \frac{\kappa_\mathrm{B}}{m} \quad \textbf{(6.2)}$$

- **Internal Energy**

$$\epsilon = c_V T \quad \textbf{(6.3)}$$

$$c_V = \frac{R}{\gamma - 1} \quad \textbf{(6.4)}$$

- **Entropy per unit mass**

$$T ds = d\epsilon + pd\left(\frac{1}{\rho}\right) \quad \textbf{(6.5)}$$

(6.1), (6.3), (6.4), (6.5) →

$$s = c_V \ln\left(\frac{p}{\rho^\gamma}\right) + s_0 \quad \textbf{(6.6)}$$

6.1 Thermal properties of a perfect gas

- **For adiabatic process**

$$\frac{ds}{dt} = 0 \quad \textbf{(6.7)} \xrightarrow{\textbf{(6.6)}} \quad \frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \quad \textbf{(6.8)}$$

- **Enthalpy per unit mass**

$$w = \epsilon + \frac{p}{\rho} \quad \textbf{(4.26)} \xrightarrow{\textbf{(6.1), (6.3)}} \quad w = \frac{\gamma}{\gamma - 1} RT \quad \textbf{(6.9)}$$

- **Bernoulli's Principle for adiabatic, perfect gas**

$$\frac{1}{2}v^2 + \int \frac{dp}{\rho} + \Phi = \text{constant} \quad \textbf{(4.38)}$$

$$dw = Tds + \frac{dp}{\rho}$$

$$\frac{1}{2}v^2 + \frac{\gamma}{\gamma - 1} RT + \Phi = \text{constant} \quad \textbf{(6.10)}$$

*** To focus on compressibility, view** $\mu = 0, K = 0$

6.2 Acoustic waves

- **Homogeneous, perfect process**

$$p_0 \quad \rho_0 \xrightarrow{\textbf{Perturbed}} \begin{array}{l} p_0 + p_1(\mathbf{x}, t) \\ \rho_0 + \rho_1(\mathbf{x}, t) \\ \mathbf{v}_1(\mathbf{x}, t) \end{array}$$

$$p_1 = c_s^2 \rho_1 \quad \textbf{(6.11)}$$

$$c_\mathrm{s} = \sqrt{\frac{dp}{d\rho}} \quad \textbf{(6.12)} \xrightarrow[\textbf{(6.8)}]{\textbf{Adiabatic}} c_\mathrm{s} = \sqrt{\frac{\gamma p_0}{\rho_0}} \quad \textbf{(6.13)}$$

- **From the Eq. of Continuity**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \xrightarrow{\textbf{Perturbed}} \frac{\partial \rho_1}{\partial t} + \nabla \cdot [(\rho_0 + \rho_1)\,\mathbf{v}_1] = 0 \quad \textbf{(6.14)}$$

Linearization

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \quad \textbf{(6.15)}$$

6.2 Acoustic waves

- **From the Euler Eq.**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{F}$$

Perturbed → $\mathbf{F} = 0$

$$(\rho_0 + \rho_1)\left[\frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla)\,\mathbf{v}_1\right] = -\nabla p_1 \quad \textbf{(6.16)}$$

(6.11) **Linearization**

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -c_s^2 \nabla \rho_1 \quad \textbf{(6.17)}$$

- **Eq. For acoustic wave**

(6.15) → **(6.17)**

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right) \rho_1 = 0 \quad \textbf{(6.18)}$$

6.2 Acoustic waves

- **Fourier components & Dispersion Relation**

Perturbation analysis is linear

Principle of superposition holds

Any perturbation → **Fourier Component**

Typical

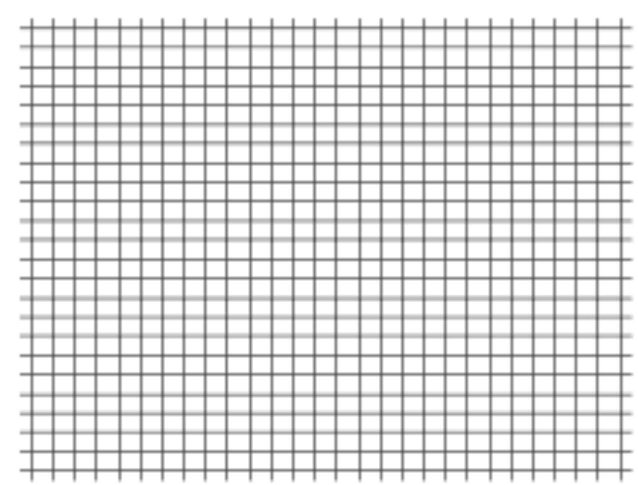
$$\rho_1 = \rho_{1,0} \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)] \quad \textbf{(6.19)}$$

Dispersion Relation

$$\omega^2 = c_s^2 k^2 \quad \textbf{(6.20)}$$

*** Phase velocity & group velocity**

$$v_p = \frac{\omega}{k}, \qquad v_g = \nabla_k \omega \quad \textbf{(6.21)}$$



6.3 Emission of acoustic waves

For comparison

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -c_s^2 \nabla \rho_1 \quad \text{(6.17)}$$

- **Fluid velocities act as sources of acoustic waves**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \longrightarrow \quad \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{(6.22)}$$

$$T_{ij} = \underline{p}\delta_{ij} + \rho v_i v_j \quad \longleftrightarrow \quad T_{ij} = \underline{p_1}\delta_{ij} + \rho v_i v_j \quad \text{(6.23)}$$

$$T_{ij} = c_s^2 \rho_1 \delta_{ij} + Q_{ij} \quad \text{(6.24)}$$

$$Q_{ij} = \rho v_i v_j + \left(p_1 - c_s^2 \rho_1\right) \delta_{ij} \quad \text{(6.25)}$$

(6.22)

$$\frac{\partial}{\partial t}(\rho v_i) + c_s^2 \frac{\partial \rho_1}{\partial x_i} = -\frac{\partial Q_{ij}}{\partial x_j} \quad \text{(6.26)}$$

6.3 Emission of acoustic waves

- **Fluid velocities act as sources of acoustic waves**

$$\frac{\partial}{\partial x_i} \quad \textbf{(6.26)} \qquad \frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho v_i\right) = 0$$

Celebrated Inhomogeneous Wave Equation

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right) \rho_1 = \frac{\partial^2 Q_{ij}}{\partial x_i \partial x_j} \qquad \textbf{(6.27)}$$

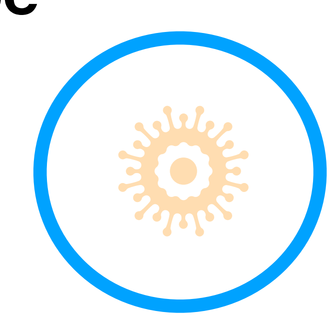
Solution

$$\rho_1(\mathbf{x}, t) = \frac{1}{c_3^2} \int \frac{\partial^2 Q_{ij}\left(\mathbf{x}', t - \left|\mathbf{x} - \mathbf{x}'\right| / c_3\right) / \partial x_i' \partial x_j'}{4\pi \left|\mathbf{x} - \mathbf{x}'\right|} dV' \qquad \textbf{(6.28)}$$

- **Acoustic wave radiation type**

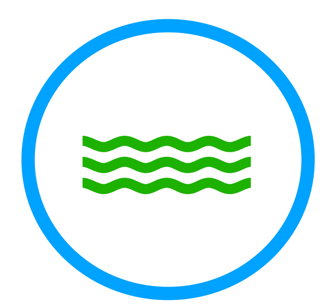
Fluid motion

Quadrupolar radiation



Obj changing volume inside

Monopole radiation



Obj oscillation inside

Dipole radiation

6.4 Steepening into shock waves & Method of characteristic

Large wave amplitude

- **Purely longitudinal 1-D wave**

Euler Eq.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \textbf{(6.29)}$$

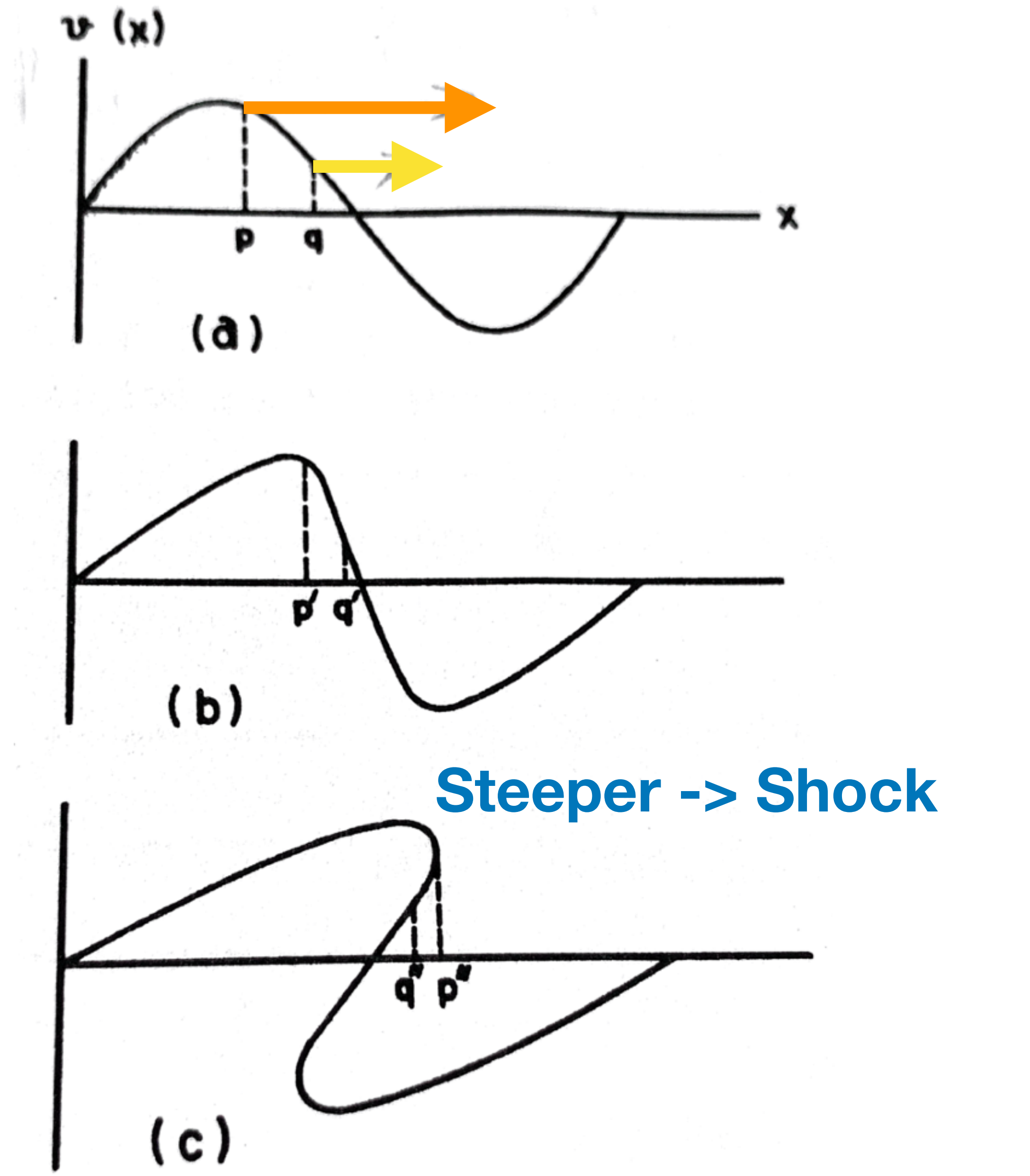
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 \quad \textbf{(6.30)}$$

Notice:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt}$$

Characteristic Curves

$$\frac{dv}{dt} = 0 \quad \textbf{(6.32)}$$



Hyperbolic PDE —**Method of Characteristics**→ **ODE**