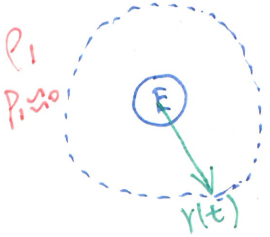
« The physics of fluids and plasmas »

Section 6.6   P118   Yuzhu



Suppose an energy E is suddenly released in an explosion producing a spherical blast wave. Let λ be a scale parameter giving the size of the blast wave at time t

Taylor-Sedov solution: $\lambda = \left(\frac{Et^2}{\rho_1}\right)^{\frac{1}{5}}$ (6.40)
(see reference)

$r(t)$: radius of a shell of gas inside the spherical blast. Here is a dimensionless distance parameter

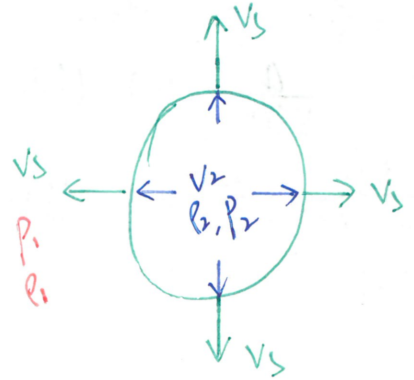
$$\xi = \frac{R}{\lambda} = R\cdot\left(\frac{\rho_1}{Et^2}\right)^{\frac{1}{5}}$$

Here I would like to derive several equations which will be used later:

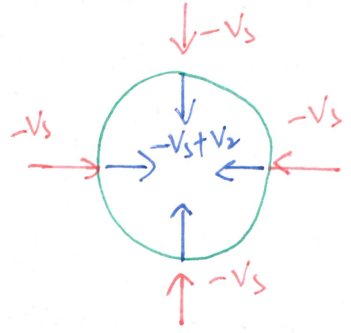
① $$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial \xi}\cdot R\cdot\left(\frac{\rho_1}{E}\right)^{\frac{1}{5}}\cdot\frac{d\,t^{-\frac{2}{5}}}{dt}$$
$$= \frac{\partial}{\partial \xi}\cdot\left(-\frac{2}{5}\right)R\cdot\left(\frac{\rho_1}{E}\right)^{\frac{1}{5}}\cdot t^{-\frac{7}{5}}$$
$$= \frac{\partial}{\partial \xi}\cdot\left(-\frac{2}{5}\right)R\cdot\left(\frac{\rho_1}{Et^2}\right)^{\frac{1}{5}}\cdot\frac{1}{t}$$
$$= -\frac{2}{5}\,\frac{\xi}{t}\,\frac{\partial}{\partial \xi}$$

② $$\frac{\partial}{\partial R} = \frac{\partial}{\partial \xi}\cdot\frac{\partial \xi}{\partial R} = \frac{\partial}{\partial \xi}\cdot\frac{\partial R}{\partial R}\cdot\left(\frac{\rho_1}{Et^2}\right)^{\frac{1}{5}}$$
$$= \frac{\xi}{R}\cdot\frac{\partial}{\partial \xi}$$

③ $$\frac{d\xi}{dt} = -\frac{2}{5}\cdot\frac{\xi}{t}, \qquad \frac{dt}{d\xi} = -\frac{5t}{2\xi}$$
$$\rightarrow \frac{d\frac{1}{t}}{d\xi} = -\frac{1}{t^2}\frac{dt}{d\xi} = \frac{5}{2\xi t}$$
$$\rightarrow \frac{d\left(\frac{1}{t}\right)^2}{d\xi} = -2\cdot\frac{1}{t^3}\frac{dt}{d\xi} = \frac{5}{t^2\xi}$$



shock is moving



in the frame of shock

In the limit of strong shock ($M_1 \to \infty$), then

① → $\rho_2 = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1$ (6.46)

④ → $\frac{-V_s+V_2}{-V_s} = \frac{\rho_1}{\rho_2} = \frac{\gamma-1}{\gamma+1}$

→ $V_2 = \frac{2}{(\gamma+1)}V_s$ (6.47)

② → $\frac{P_2}{P_1} = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1} = \frac{2}{P_1(\gamma+1)}\rho_1 V_s^2$

$M_1^2 = \frac{V_s^2}{\gamma P_1/\rho_1} = \frac{\rho_1 V_s^2}{\gamma P_1}$, when $M_1 \to \infty$,

$2\gamma M_1^2-(\gamma-1) \approx 2\gamma M_1^2 = 2\gamma\cdot\frac{\rho_1 V_s^2}{\gamma P_1} = \frac{2\rho_1 V_s^2}{P_1}$

$\rightarrow \quad p_2 = \frac{2}{\gamma+1}\rho_1 V_s^2 \qquad (6.48)$

Now introduce the dimensionless variables $\rho'(\xi)$, $v'(\xi)$ and $p'(\xi)$ in the following manner:

$\circ \quad \rho(R,t) = \rho_2 \rho'(\xi) = \rho_1 \frac{\gamma+1}{\gamma-1}\rho'(\xi) \qquad (6.49)$

$\circ \quad v(R,t) = V_2 \cdot \frac{R}{R_s} v'(\xi) = V_2 \cdot \frac{R}{t} \cdot \frac{1}{\frac{R_s}{t}} v'(\xi)$

According to equation 6.43, $V_s = \frac{2}{5}\cdot\frac{R_s}{t}$, then

$$v(R,t) = \frac{R}{t}\cdot\frac{V_2}{\frac{5}{2}V_s} v'(\xi) = \frac{4}{5(\gamma+1)} v'(\xi) \qquad (6.50)$$

$\circ \quad p(R,t) = p_2\left(\frac{R}{R_s}\right)^2 p'(\xi) = \frac{8\rho_1}{25(\gamma+1)}\left(\frac{R}{t}\right)^2 p'(\xi) \qquad (6.51)$

where $\rho'(\xi_0) = v'(\xi_0) = p'(\xi) = 1 \qquad (6.52)$

The continuity equation:

$$\frac{\partial}{\partial t}(\text{density}) + \operatorname{div}(\text{its flux}) = 0 \qquad (4.22)$$

Euler equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (6.29)$$

adiabatic equation:

$$\frac{d}{dt}\ln\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad (6.8)$$

According to the properties of partial differentiation

$$\frac{d}{dt}\ln\left(\frac{p}{\rho^\gamma}\right) = \frac{\partial}{\partial t}\ln\left(\frac{p}{\rho^\gamma}\right) + \frac{\partial}{\partial R}\left[\ln\left(\frac{p}{\rho^\gamma}\right)\right]\cdot\frac{dR}{dt}$$
$$= \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial R}\right)\ln\left(\frac{p}{\rho^\gamma}\right) = 0$$

$\rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{1}{R^2}\frac{\partial}{\partial R}(R^2\rho v) = 0 \qquad (6.53)$

$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} \qquad (6.54)$

$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial R}\right)\ln\frac{p}{\rho^\gamma} = 0 \qquad (6.55)$

6.56, 6.57, 6.58 derivation process:

$\boxed{6.53} \rightarrow \boxed{6.56}$

$$\frac{\partial \rho}{\partial t} + \frac{1}{R^2}\cdot\frac{\partial}{\partial R}(R^2\rho v) = 0$$

$\rightarrow -\frac{2}{5}\cdot\frac{\xi}{t}\frac{d}{d\xi}\left[\rho_1\frac{\gamma+1}{\gamma-1}\rho'(\xi)\right] + \cancel{\frac{1}{R^2}}\cdot\frac{\xi}{\cancel{R}}\cdot\frac{d}{d\xi}\left[\cancel{R^2}\rho_1\frac{\gamma+1}{\gamma-1}\rho'\cdot\frac{4\cancel{R}v'}{5(\cancel{\gamma+1})t}\right] = 0$

$\rightarrow -\frac{2}{5}\frac{\xi}{t}\cdot\rho_1\frac{\gamma+1}{\gamma-1}\cdot\frac{d\rho'}{d\xi} + \frac{4}{5}\rho_1\cdot\frac{\xi}{\gamma-1}\frac{d}{d\xi}\left(\frac{\rho'v'}{t}\right) = 0$

$\rightarrow -\frac{\xi}{t}\frac{d\rho'}{d\xi} + \frac{2\xi}{\gamma+1}\left\{\left[\frac{d}{d\xi}\left(\frac{1}{t}\right)\right]\cdot\rho'v' + \frac{1}{t}\frac{d}{d\xi}(\rho'v')\right\} = 0$

$$\frac{d}{d\xi}\left(\frac{1}{t}\right) = \frac{5}{2\xi t}$$

$\rightarrow -\frac{\xi}{t}\left(\frac{d\rho'}{d\xi}\right) + \frac{2\xi}{\gamma+1}\left[\frac{5}{2\xi t}\rho'v' + \frac{1}{t}\frac{d}{d\xi}(\rho'v')\right] = 0$

$\rightarrow -\xi\frac{d\rho'}{d\xi} + \frac{2}{\gamma+1}\left[\boxed{\frac{5}{2}}\rho'v' + \xi\frac{d}{d\xi}(\rho'v')\right] = 0 \qquad (6.56)$

?

$\boxed{6.54} \rightarrow \boxed{6.57}$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial R}$$

$\rightarrow$ Left side $= -\frac{2}{5}\cdot\frac{\xi}{t}\frac{d}{d\xi}\left[\frac{4}{5(\gamma+1)}\cdot\frac{R}{t}v'\right]$

$$+\frac{4}{5(r+1)}\cdot\frac{R}{t}v'\cdot\frac{\xi}{R}\cdot\frac{d}{d\xi}\left[\frac{4}{5(r+1)}\cdot\frac{R}{t}v'\right]$$

$$=-\frac{8}{25}\cdot\frac{\xi R}{t(r+1)}\cdot\frac{d(\frac{v'}{t})}{d\xi}+\frac{16}{25}\cdot\frac{R\xi v'}{t(r+1)^2}\cdot\frac{d(\frac{v'}{t})}{d\xi}$$

$$=-\frac{8}{25}\cdot\frac{\xi R}{t(r+1)}\left(1-\frac{2v'}{r+1}\right)\left[v'\frac{d(\frac{1}{t})}{d\xi}+\frac{1}{t}\cdot\frac{dv'}{d\xi}\right]$$

$$=-\frac{8}{25}\frac{\xi R}{t(r+1)}\left(1-\frac{2v'}{r+1}\right)\left(\frac{5v'}{2\xi t}+\frac{1}{t}\cdot\frac{dv'}{d\xi}\right)$$

$\longrightarrow$ Right side $=-\frac{r-1}{\rho_1(r+1)\rho'}\cdot\frac{\xi}{R}\cdot\frac{d}{d\xi}\left[\frac{8\rho_1}{25(r+1)}\cdot\left(\frac{R}{t}\right)^2p'\right]$

$$=-\frac{8}{25}\frac{r-1}{(r+1)^2}\cdot\frac{R\xi}{\rho'}\cdot\frac{d}{d\xi}\left[\left(\frac{1}{t}\right)^2\cdot p'\right]$$

$$=-\frac{8}{25}\frac{r-1}{(r+1)^2}\cdot\frac{R\xi}{\rho'}\left[p'\cdot\frac{d}{d\xi}\left(\frac{1}{t}\right)^2+\frac{1}{t^2}\frac{dp'}{d\xi}\right]$$

$$\frac{d}{d\xi}\left(\frac{1}{t}\right)^2=\frac{5}{t^2\xi}$$

$$=-\frac{8}{25}\frac{r-1}{(r+1)^2}\cdot\frac{R\xi}{\rho'}\left(\frac{5p'}{t^2\xi}+\frac{1}{t^2}\cdot\frac{dp'}{d\xi}\right)$$

eliminate $\frac{4}{5}\cdot\frac{R}{t^2(r+1)}$ for both sides:

$$-\frac{2}{5}\xi\left(1-\frac{2v'}{r+1}\right)\left(\frac{5v'}{2\xi}+\frac{dv'}{d\xi}\right)=-\frac{2}{5}\frac{r-1}{r+1}\frac{1}{\rho'}\left(5p'+\xi\frac{dp'}{d\xi}\right)$$

$$-v'-\frac{2}{5}\xi\frac{dv'}{d\xi}+\frac{4}{5}\cdot\frac{1}{r+1}\left(\boxed{\frac{5}{2}}v'^2+v'\frac{dv'}{d\xi}\right)=-\frac{2}{5}\frac{r-1}{r+1}\cdot\frac{1}{\rho'}\left(\boxed{5}p'+\xi\frac{dp'}{d\xi}\right)=0$$

(6.57)

$\boxed{6.55}\rightarrow\boxed{6.58}$?

$$\left\{\begin{array}{ll}\frac{d\ln x}{dx}=\frac{1}{x} & ① \\ \ln A\cdot B=\ln A+\ln B & ② \end{array}\right.$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial R}\right)\ln\frac{P}{\rho^r}=0$$

$$\rightarrow\left[-\frac{2}{5}\frac{\xi}{t}\frac{d}{d\xi}+\frac{4v'}{5(r+1)}\cdot\cancel{\frac{R}{t}}\cdot\cancel{\frac{\xi}{R}}\cdot\frac{d}{d\xi}\right]\left[\ln\frac{\frac{8\rho_1}{25(r+1)}\cdot\left(\frac{R}{t}\right)^2p'}{\left(\rho_1\frac{r+1}{r-1}\rho'\right)^r}\right]=0$$

$$\rightarrow\left[-\frac{2}{5}\boxed{\frac{\xi}{t}}\frac{d}{d\xi}+\frac{4v'}{5(r+1)}\cdot\boxed{\frac{\xi}{t}}\frac{d}{d\xi}\right]\left\{\underbrace{\ln\left[\frac{\frac{8\rho_1}{25(r+1)}}{\left(\rho_1\frac{r+1}{r-1}\right)^r}\cdot\frac{R^2}{t^2}\cdot\frac{p'}{\rho'^r}\right]}_{②}\right\}=0$$

$$\rightarrow\left[-\frac{2}{5}\frac{\xi}{t}\cdot\frac{d}{d\xi}+\frac{4v'}{5(r+1)}\cdot\frac{\xi}{t}\frac{d}{d\xi}\right]\left[\underbrace{\ln\frac{\frac{8\rho_1}{25(r+1)}}{\left(\rho_1\frac{r+1}{r-1}\right)^r}}_{\text{constant}}+2\ln R-2\ln t+\ln\left(\frac{p'}{\rho'^r}\right)\right]=0$$

$$\frac{d\ln R}{d\xi}=\frac{\frac{1}{R}dR}{d\xi}=\frac{1}{\xi}$$

$$\frac{d\ln t}{d\xi}=\frac{\frac{1}{t}\ln t}{d\xi}=\frac{1}{t}\cdot\left(-\frac{5}{2}\cdot\frac{t}{\xi}\right)=-\frac{5}{2}\cdot\frac{1}{\xi}$$

$$\frac{d(1+x)}{dx}=1,$$

$$\rightarrow\left(1-\frac{2v'}{r+1}\right)\frac{d}{d\xi}\left[2\ln R-2\ln t+\ln\left(\frac{p'}{\rho'^r}\right)\right]=0$$

$$\rightarrow\frac{r+1-2v'}{r+1}\cdot\left[\frac{2}{\xi}+\frac{5}{\xi}+\frac{d}{d\xi}\ln\left(\frac{p'}{\rho'^r}\right)\right]=0$$

$\rightarrow$

Aim to solve the Coupled nonlinear ordinary differential equations (6.56-6.58) with the boundary condition

$$\rho'(\xi_0) = v'(\xi_0) = p'(\xi_0) = 1 \quad (6.52)$$

to be satisfied at ξ_0.

The value of ξ_0 can be obtained from the condition that the total energy of the blast wave remains constant, i.e.

$$E = \int_0^{R_s} \Big(\underbrace{\boxed{\frac{\rho v^2}{2}}}_{\text{kinetic energy}} + \underbrace{\boxed{\frac{p}{\gamma-1}}}_{\text{thermal energy}} \Big) 4\pi R^2 dR$$

the thermal energy density $C_v \rho T = \frac{p}{\gamma-1}$

$$\xi = R\left(\frac{\rho_1}{Et^2}\right)^{\frac{1}{5}} \quad (6.41)$$

$$R_s = \xi_0 \left(\frac{Et^2}{\rho_1}\right)^{\frac{1}{5}} \quad (6.42)$$

$$dR = \left(\frac{\rho_1}{Et^2}\right)^{-\frac{1}{5}} d\xi$$

$$\rightarrow \int_0^{\xi_0} \left(\frac{\rho_1 \frac{\gamma+1}{\gamma-1} \rho' \cdot \left(\frac{4}{5(\gamma+1)} \cdot \frac{R}{t} v'\right)^2}{2} + \frac{\frac{8\rho_1}{25(\gamma+1)} \cdot \left(\frac{R}{t}\right)^2 p'}{\gamma-1} \right)$$

$$\times 4\pi \times \frac{\xi}{\left(\frac{\rho_1}{Et^2}\right)^{\frac{2}{5}}} d\xi = E = \left[R\left(\frac{\rho_1}{Et^2}\right)^{\frac{1}{5}}\right]^2 \cdot E$$

$$\rightarrow \int_0^{\xi_0} \frac{32\pi}{25(\gamma^2-1)} \cdot (\rho' v'^2 + p') \cdot \frac{\xi^2 \boxed{R^2 \rho_1^{\frac{2}{5}} E^{\frac{3}{5}}}}{\boxed{t^{\frac{4}{5}}}} d\xi = E$$

$$\rightarrow \frac{32\pi}{25(\gamma^2-1)} \int_0^{\xi_0} (\rho' v'^2 + p') \xi^4 d\xi = 1 \quad (6.59)$$

From which ξ_0 can be found.

$\gamma = 1.4$ is appropriate for air.