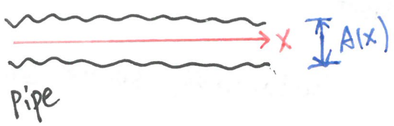
《The physics of fluids and plasmas》

Section 6.7 P122 Xuzhu



Consider a steady, adiabatic gas flow ① not vary in time
② satisfy adiabatic relation

$$\rightarrow \frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = \underbrace{\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right)}_{=0} + \frac{\partial}{\partial x}\cdot\left(\frac{p}{\rho^{\gamma}}\right)\frac{dx}{dt}$$

$$= v\cdot\frac{d}{dx}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

namely $\frac{d}{dx}\left(\frac{p}{\rho^{\gamma}}\right) = 0$

$$\rightarrow \rho^{-\gamma}\frac{dp}{dx} = \gamma\cdot\rho^{-\gamma-1}p\cdot\frac{d\rho}{dx}$$

$$\rightarrow \frac{dp}{dx} = \frac{\gamma p}{\rho}\cdot\frac{d\rho}{dx}$$

$$\rightarrow \frac{dp}{dx} = c_s^2\frac{d\rho}{dx} \qquad (6.60)$$

where $c_s = \sqrt{\frac{\gamma p}{\rho}}$ (6.61)

x is the only independent variable in this problem. Since the same mass flux has to pass through the pipe at any x, we must have

$$\rho(x)\,v(x)\,A(x) = \text{Constant}$$

The Euler equation: $\underbrace{\frac{\partial v}{\partial t}}_{=0} + v\frac{dv}{dx} = -\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}$ (6.29)

$$\rightarrow v\frac{dv}{dx} = -\frac{c_s^2}{\rho}\cdot\frac{d\rho}{dx} \qquad (6.63)$$

Differentiating (6.62):

$$\frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{A}\cdot\frac{dA}{dx} = 0 \qquad (6.64)$$

$$\rightarrow \frac{1}{\rho}\cdot\frac{d\rho}{dx} = -\frac{v}{c_s^2}\frac{dv}{dx}$$

$$\rightarrow -\frac{v}{c_s^2}\frac{dv}{dx} + \frac{1}{v}\frac{dv}{dx} = -\frac{1}{A}\frac{dA}{dx}$$

Important

$$\left(1-\frac{v^2}{c_s^2}\right)\frac{1}{v}\frac{dv}{dx} = -\frac{1}{A}\frac{dA}{dx}$$

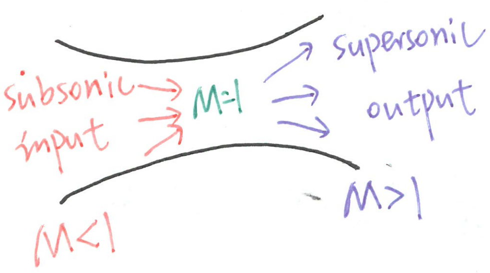
$$\boxed{(1-M^2)\cdot\frac{1}{v}\cdot\frac{dv}{dx} = -\frac{1}{A}\frac{dA}{dx} \qquad (6.65)}$$

Where $M = \frac{v}{c_s}$ is the local Mach number at any point along the pipe.

Case 1:
when $M<1$: subsonic, $\frac{dv}{dx}$ and $\frac{dA}{dx}$ in (6.65) have opposite signs.

If a narrowing of the pipe ($dA\downarrow$), the flow will become faster ($dv\uparrow$).

Case 2:
When $M>1$: supersonic, $\frac{dv}{dx}$ and $\frac{dA}{dx}$ in 6.65) have same signs. namely $\Delta A\uparrow \rightarrow \Delta v\uparrow$



Case 3: transition from

$$M<1 \Rightarrow M=1 \Rightarrow M>1$$

$$\Downarrow$$

$$\frac{dA}{dx}=0$$

namely the variation of area is zero.

This pipe is called a de Laval nozzle.

In order to get this kind of transition, the pressure boundary conditions at the two sides have to be adjusted.

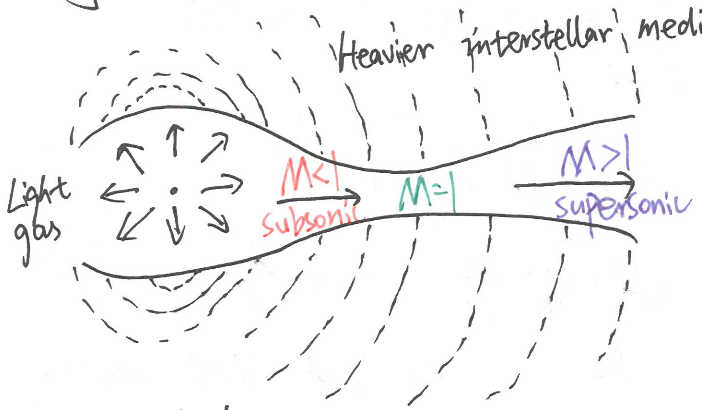


Figure 6.8

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Consider a steady spherical flow: ① velocity v is independent of time ② in the radial direction (+/−)

Same mass flux has to flow through spherical surfaces at different distances.

$$r^2\rho v = \text{constant} \Rightarrow \ln(r^2\rho v) = \text{constant}$$

Differentiating:

$$2r + \frac{1}{\rho}\cdot\frac{d\rho}{dr} + \frac{1}{v}\frac{dv}{dr} = 0 \qquad (6.66)$$

Euler equation gives:

$$\rho v\frac{dv}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho \qquad (6.67)$$

Inward: steady spherical accretion

Outward: steady spherical accretion

} same in the view of mathematic

Isothermal: $p = R\rho T$, T is constant.

$p = V_c^2\rho = RT\rho$, V_c is the isothermal sound speed

$$(6.66) \rightarrow \boxed{\frac{1}{\rho}\frac{d\rho}{dr}} = -\left(\frac{1}{v}\frac{dv}{dr} + \frac{2}{r}\right)$$

$$(6.67) \rightarrow \rho v\frac{dv}{dr} = -\frac{V_c^2 d\rho}{dr} - \frac{GM}{r^2}\rho$$

$$\xrightarrow{\text{eliminate } \rho} v\cdot\frac{dv}{dr} = -V_c^2\cdot\boxed{\frac{1}{\rho}\frac{d\rho}{dr}} - \frac{GM}{r^2}$$

$$v\cdot\frac{dv}{dr} = V_c^2\left(\frac{1}{v}\frac{dv}{dr} + \frac{2}{r}\right) - \frac{GM}{r^2}$$

$$\left(v - \frac{V_c^2}{v}\right)\frac{dv}{dr} = \frac{2V_c^2}{r} - \frac{GM}{r^2} \qquad (6.68)$$

Then let $v = V_c$, the distance should be

$$r = r_c = \frac{GM}{2V_c^2}$$

Integrate (6.68)

$$\int\left(v - \frac{V_c^2}{v}\right)dv = \int\left(\frac{2V_c^2}{r} - \frac{GM}{r^2}\right)dr$$

$$\rightarrow \left(\frac{v^2}{2} - \frac{V_c^2}{2}\log v^2\right) = 2\log r + \frac{GM}{r} + C$$

divide $V_c^2/2$

$$\left(\frac{v}{V_c}\right)^2 - \log\left(\frac{v}{V_c}\right)^2 = 4\log\frac{r}{r_c} + \frac{2GM}{rV_c^2} + C \qquad (6.69)$$

See Fig 6.9.